Myopically Verifiable Probabilistic Certificate for Long-term Safety

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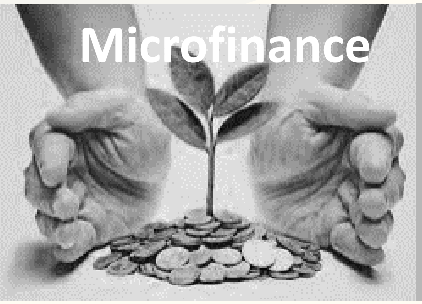
理学

Stochastic safe control
Robust control
Optimization
Information theory …

科学

Neuroscience
Biomolecular control…



工学

Safety is critical for intelligent systems

Autonomous
vehicles

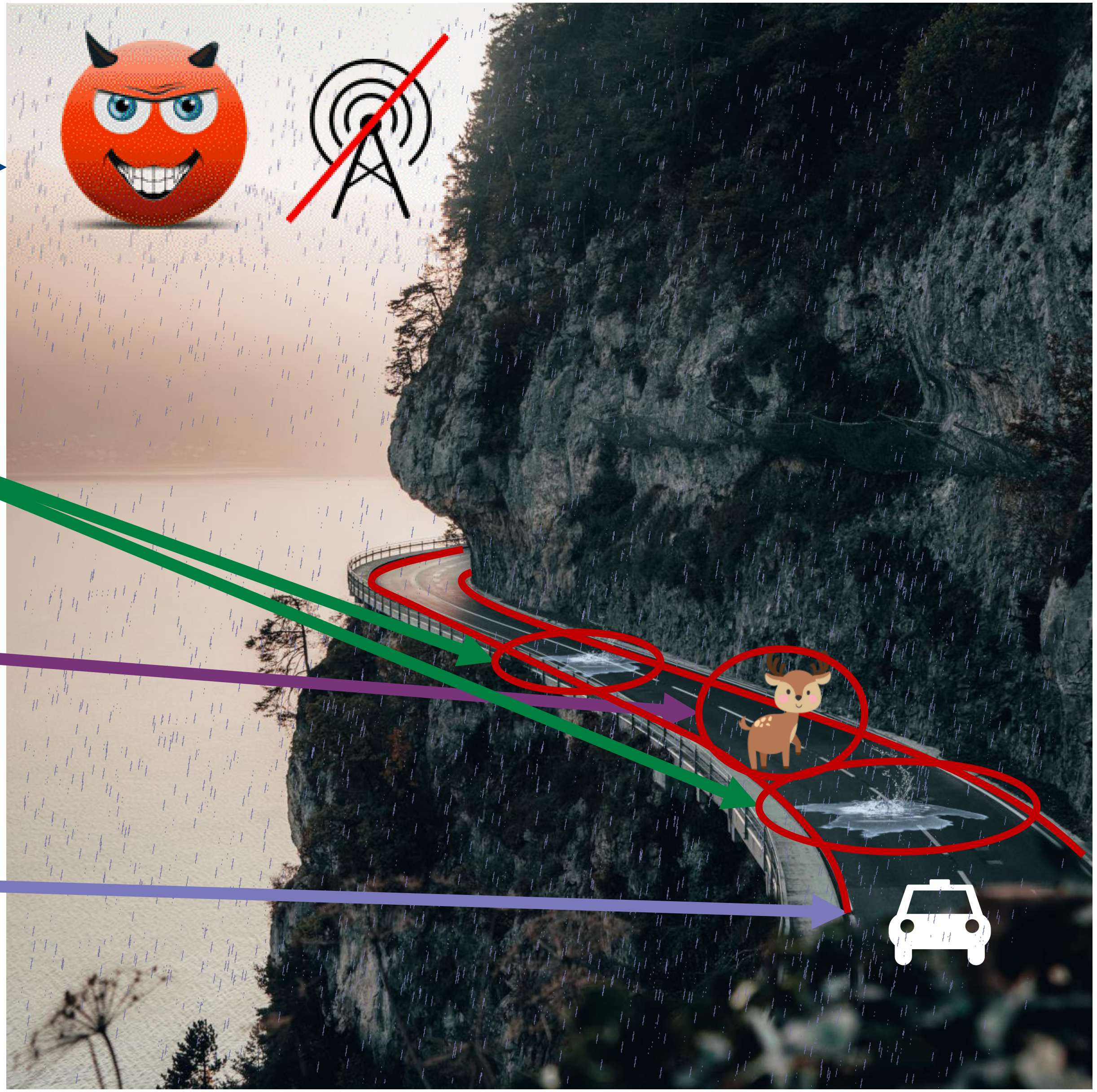
Cobots
Intelligent manufacturing

Drones

Safety is critical for intelligent systems

Dealing with uncertainty

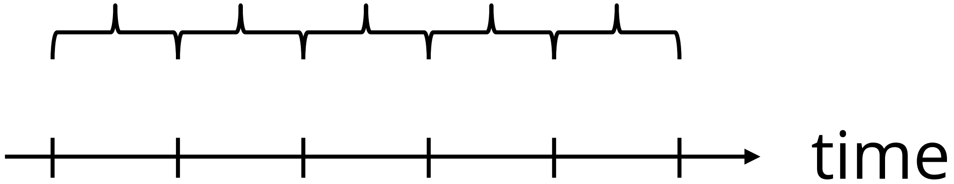
- Resilience
- Adaptability
- Collision Avoidance
- Regular Operation



Challenges

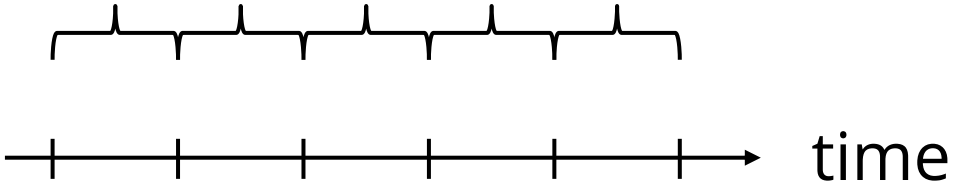
Deterministic system

safe at each step → safe at all time

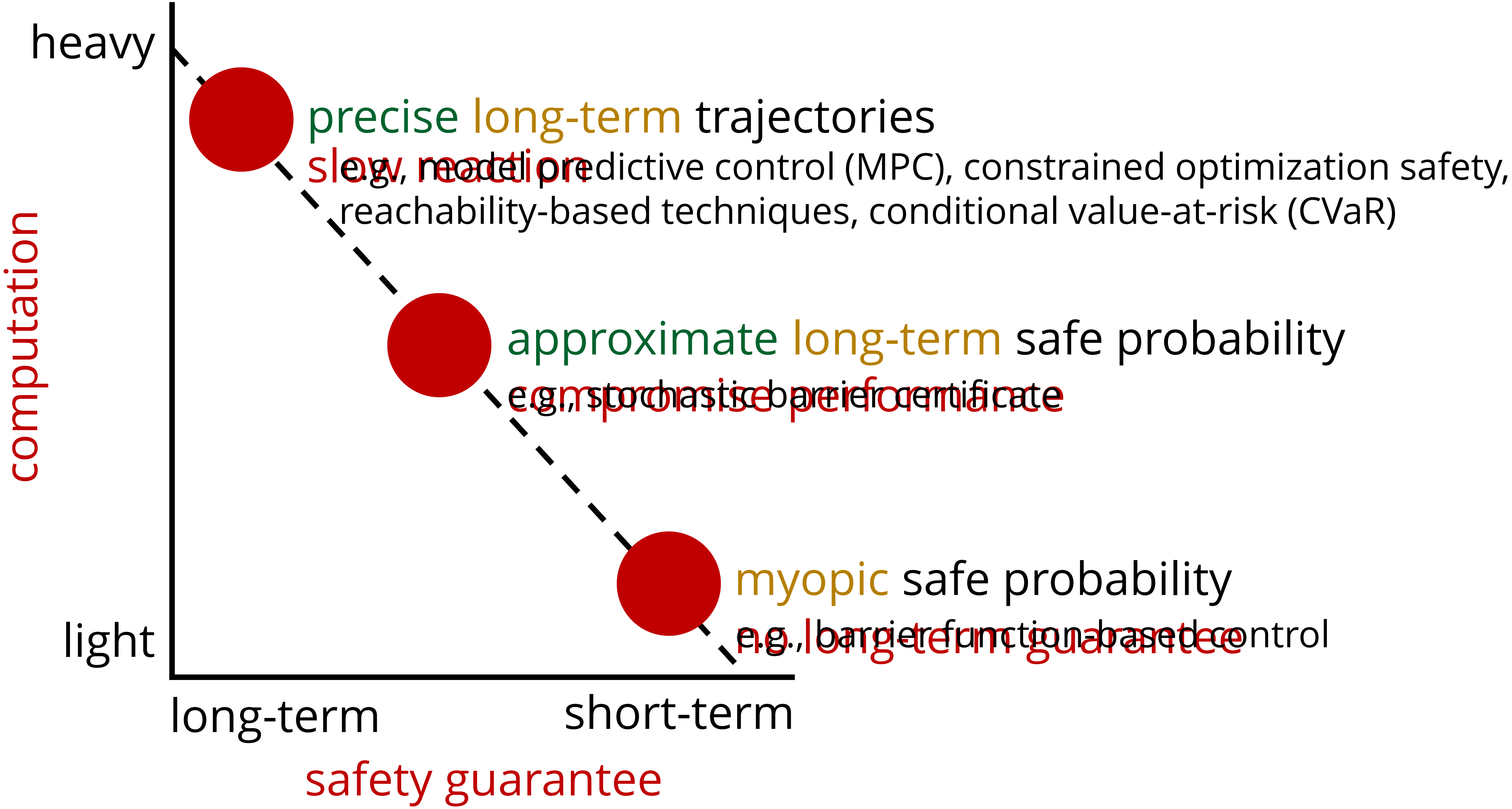


Stochastic system

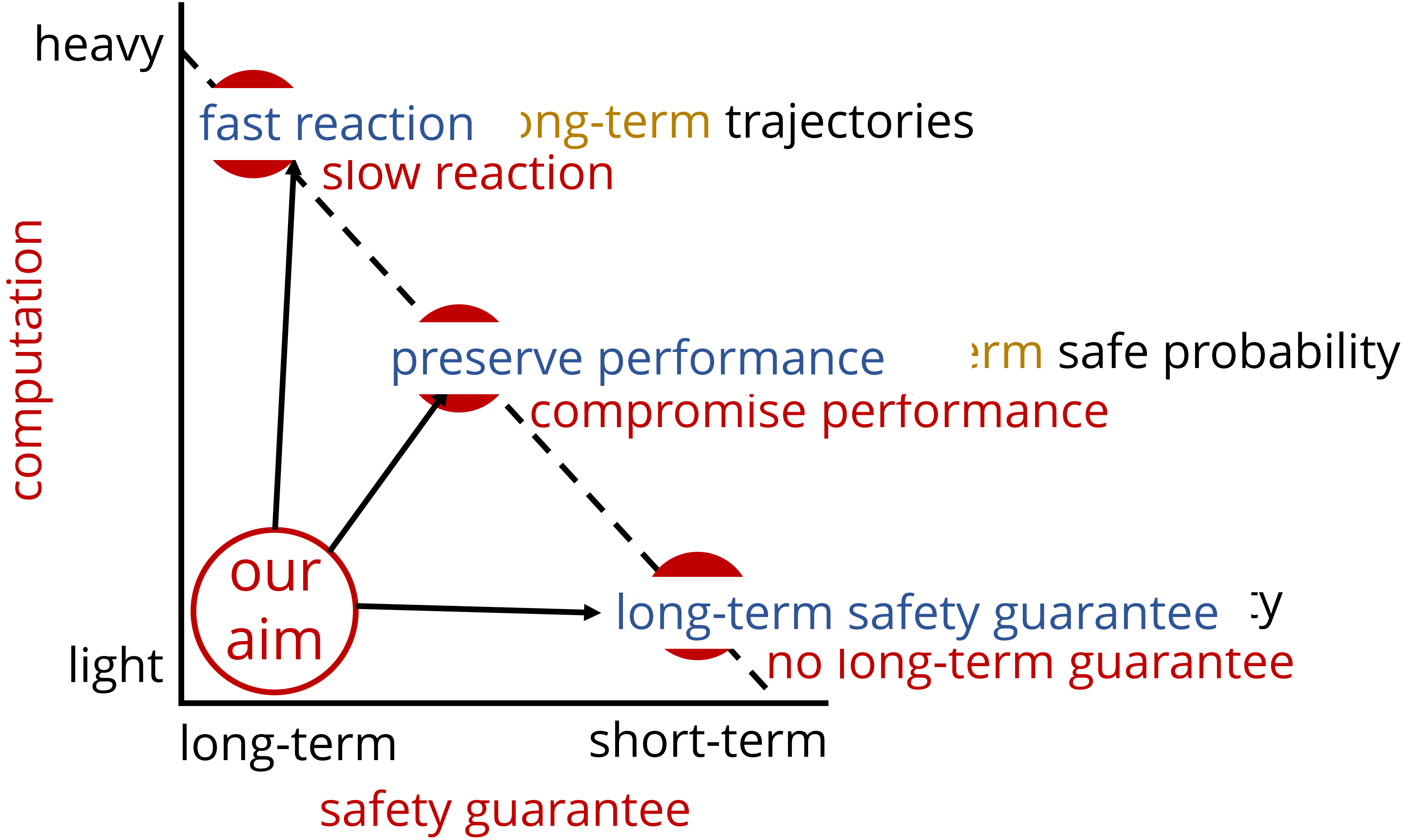
safe at each step with probability $1-\delta$ → Long-term safe probability can scale according to $\lim_{t\to\infty}(1-\delta)^t = 0$



Challenges


heavy
light
computation
long-term
short-term
safety guarantee
precise long-term trajectories
slow reaction
e.g., model predictive control (MPC), constrained optimization safety,
reachability-based techniques, conditional value-at-risk (CVaR)
approximate long-term safe probability
compromise performance
e.g., stochastic barrier certificate
myopic safe probability
no long-term guarantee
e.g., barrier function-based control

Challenges

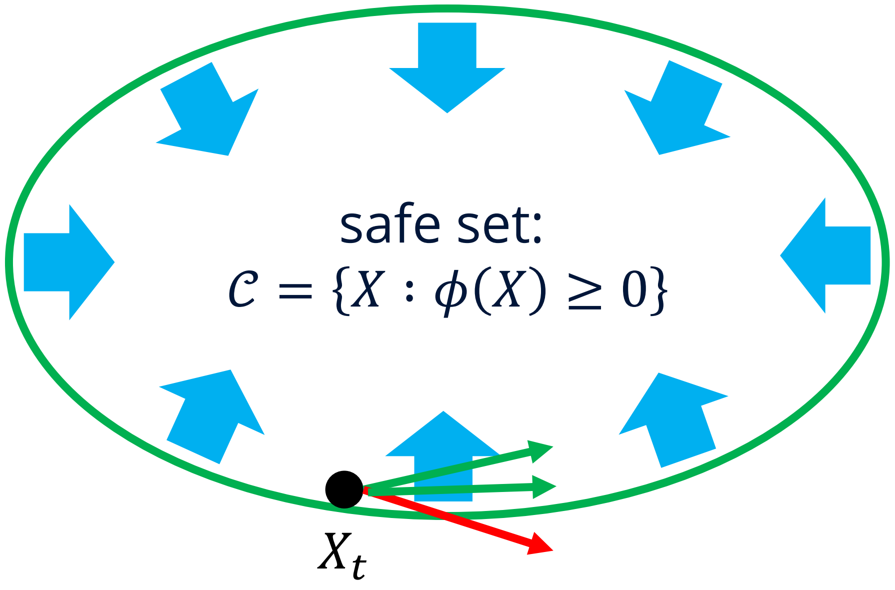


heavy
fast reaction
ong-term trajectories
slow reaction
computation
preserve performance
erm safe probability
compromise performance
our aim
long-term safety guarantee
y
no long-term guarantee
light
long-term
short-term
safety guarantee

Proposed Method: Intuitions

Imposing forward invariance on

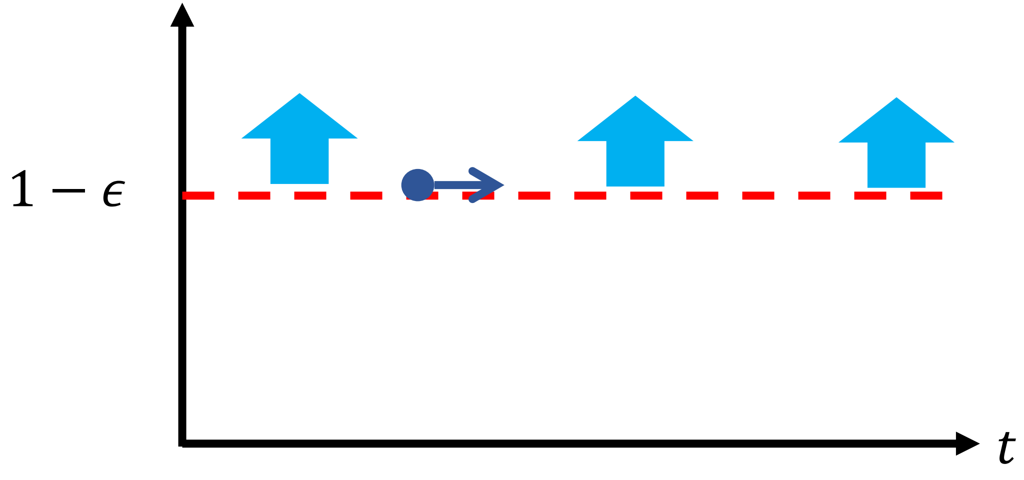
State space



Tail probability can accumulate over time

Probability space

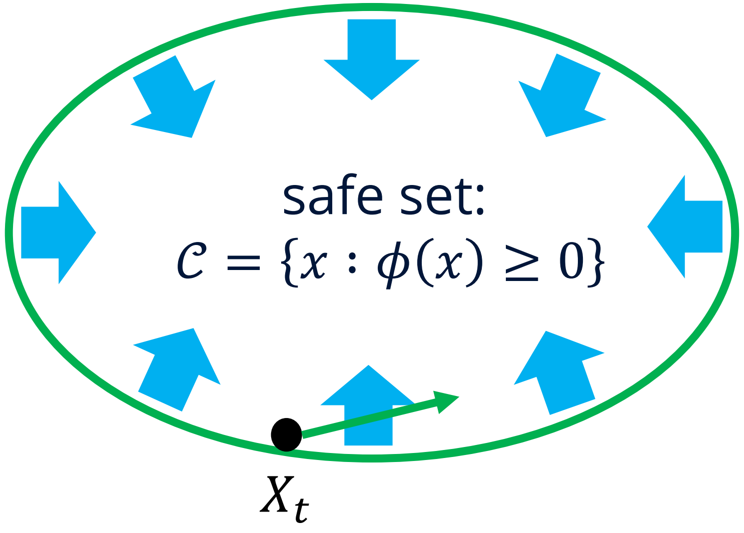
Long-term safety probability
$\Pr(X_\tau \in \mathcal{C}, \tau \in [t, t+T])$



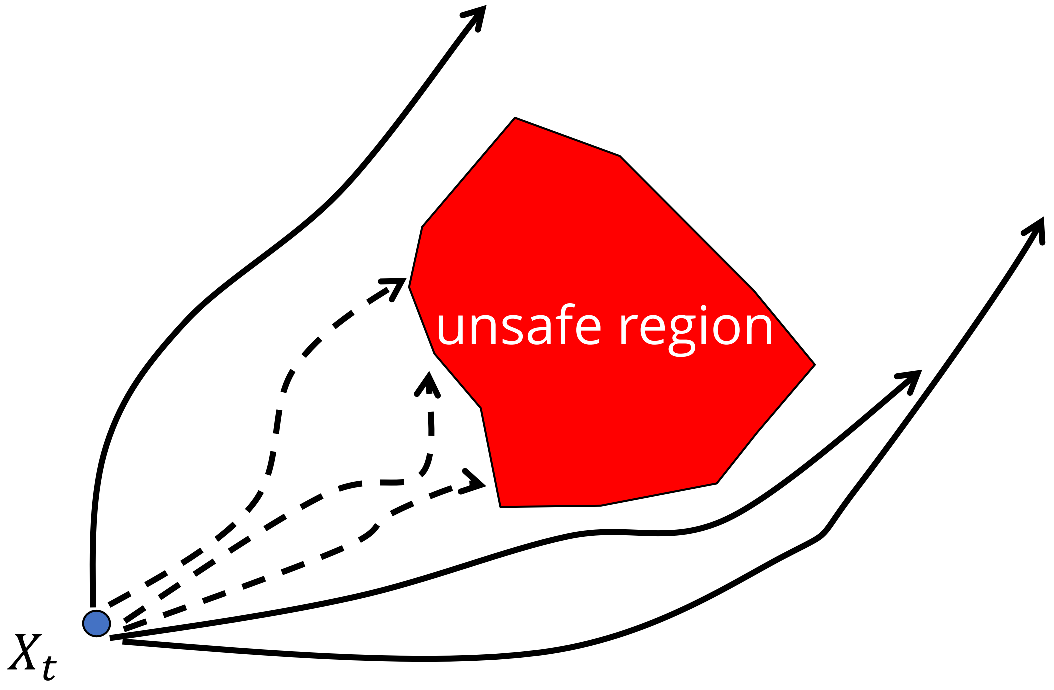
Direct control over accumulation of tail events

Proposed Method: Intuition

Barrier function based
-> Myopic evaluation



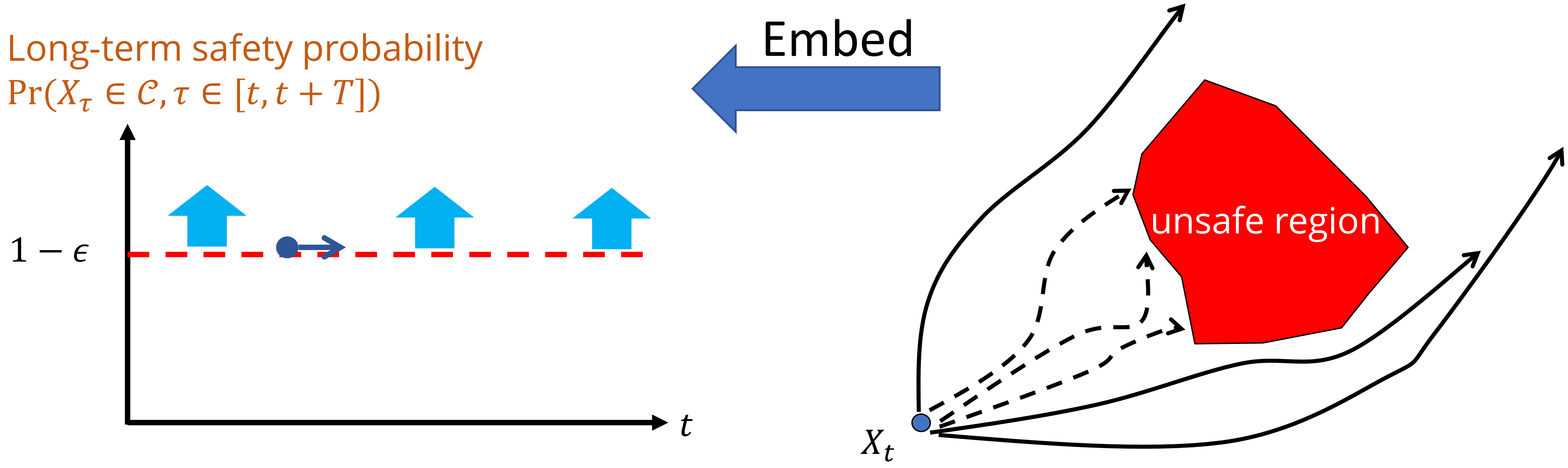
Reachability based
-> Ensures long-term safety



Proposed Method: Intuition

Barrier function based
-> Myopic evaluation

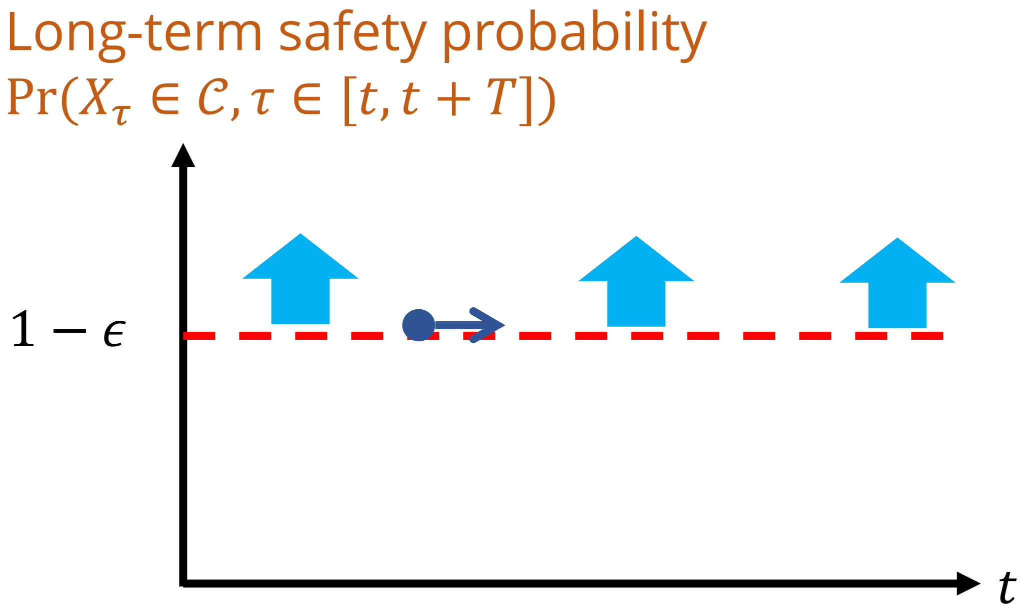
Reachability based
-> Ensures long-term safety



Direct control over
accumulation of tail events

Proposed Method

$$\boldsymbol{F}(X_t) = \Pr(X_\tau \in \mathcal{C}, \tau \in [t, t+T] | X_t)$$



Direct control over accumulation of tail events

$$A\boldsymbol{F}(X_t) \geq -\alpha(\boldsymbol{F}(X_t) - (1-\epsilon))$$

time derivative of safety probability

desired safety probability

A: infinitesimal generator

α: monotonically increasing, concave, $\alpha(0) \leq 0$

Theoretical Guarantees

Theorem: Given

$$\boldsymbol{F}(X_0) > 1 - \epsilon,$$

if we choose the control action to satisfy

$$A\boldsymbol{F}(X_t) \geq -\alpha(\boldsymbol{F}(X_t) - (1 - \epsilon)) \text{ for } t > 0,$$

then we have

$$\Pr(X_\tau \in \mathcal{C}, \tau \in [t, t + T]) \geq 1 - \epsilon \text{ for } \forall t > 0$$

$\alpha: \mathbb{R} \to \mathbb{R}$ is a monotonically increasing concave function that satisfies $\alpha(0) \leq 0$.

Proposed Safety Condition

$$A\boldsymbol{F}(X_t) \geq -\alpha(\boldsymbol{F}(X_t) - (1-\epsilon))$$

$$dX_t = (f(X_t) + g(X_t)U_t)dt + \sigma(X_t)dW$$

Affine control

$$\mathcal{L}_f\boldsymbol{F}(X_t) + \left(\mathcal{L}_g\boldsymbol{F}(X_t)\right)\boldsymbol{U_t} + \frac{1}{2}\mathrm{tr}([\sigma(X_t)]^\intercal \mathrm{Hess}\boldsymbol{F}(X_t)[\sigma(X_t)]) \geq -\alpha(\boldsymbol{F}(X_t) - (1-\epsilon))$$

linear constraints of U_t

Simulation

system dynamic: $$dx_t = (2x_t + 2.5u_t)\,dt + 2dw_t$$

initial state: $$x_0 = 3$$

safe set: $$\mathcal{C} = \{x \in \mathbb{R} : x - 1 > 0\}$$

nominal controller: $$N(x_t) = 2.5x_t$$

desired safety probability: $$1 - \epsilon = 0.9$$

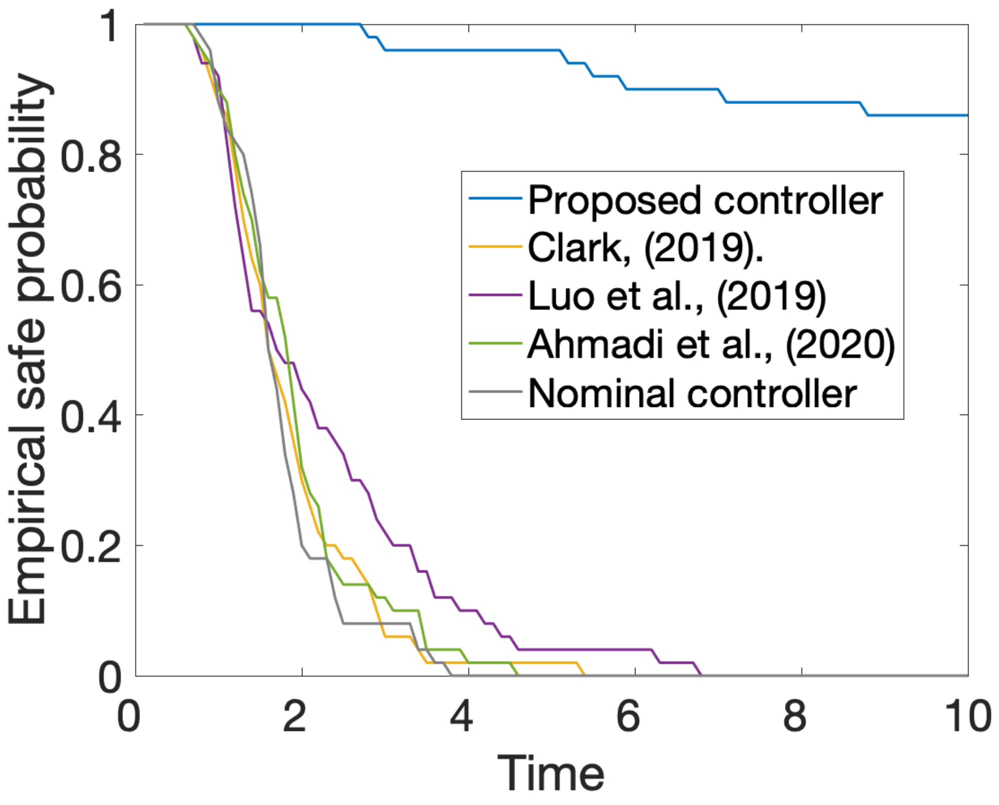
Simulation

Proposed: $A\boldsymbol{F}(X_t) \geq -\alpha(\boldsymbol{F}(X_t) - (1-\epsilon))$

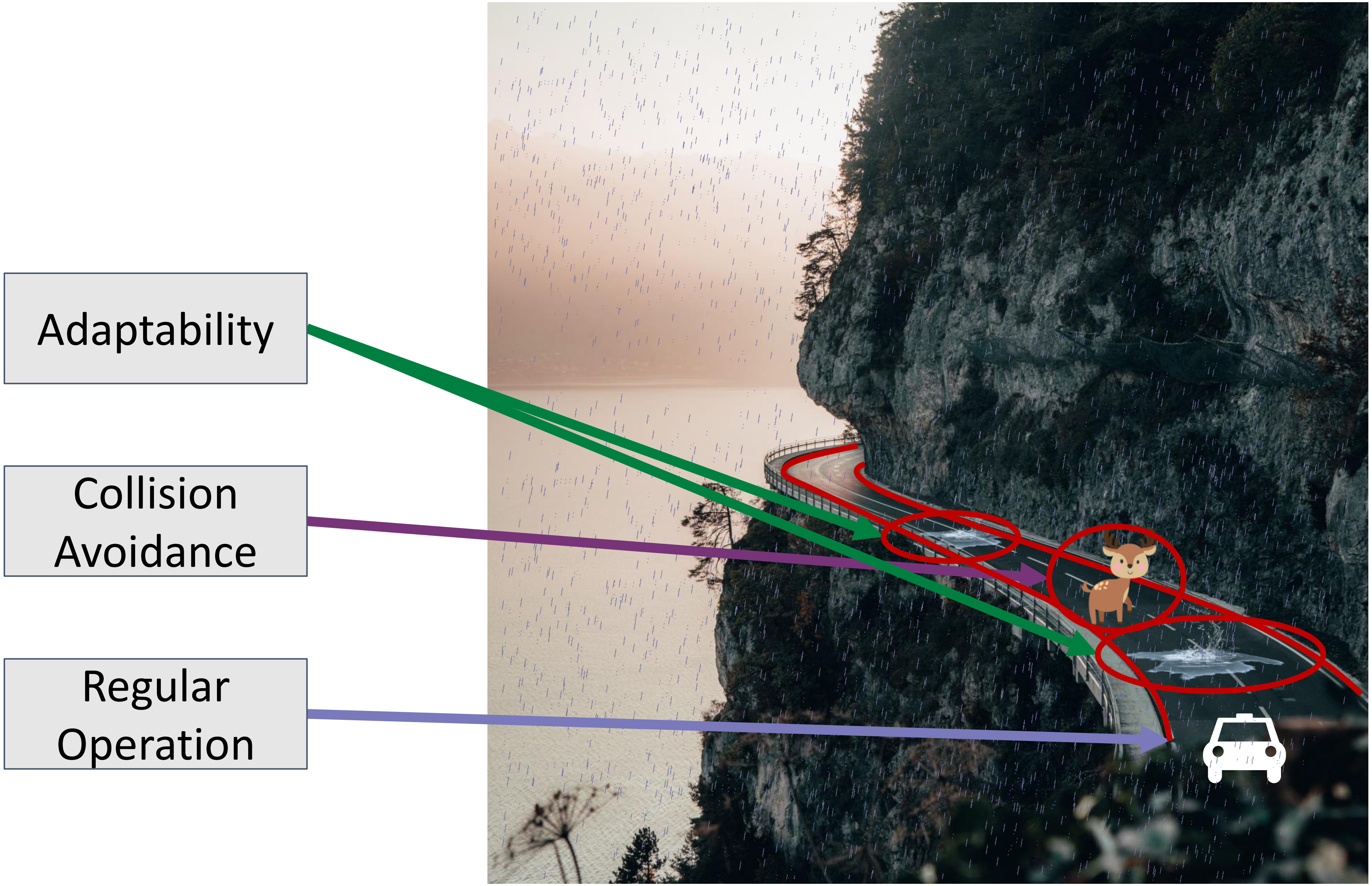
Clark: $A\phi(X_t) \geq -\alpha\phi(X_t)$

Luo et al.: $\mathbb{P}(d\phi(X_t, U_t) + \alpha\phi(X_t) \geq 0) \geq 1 - \epsilon$

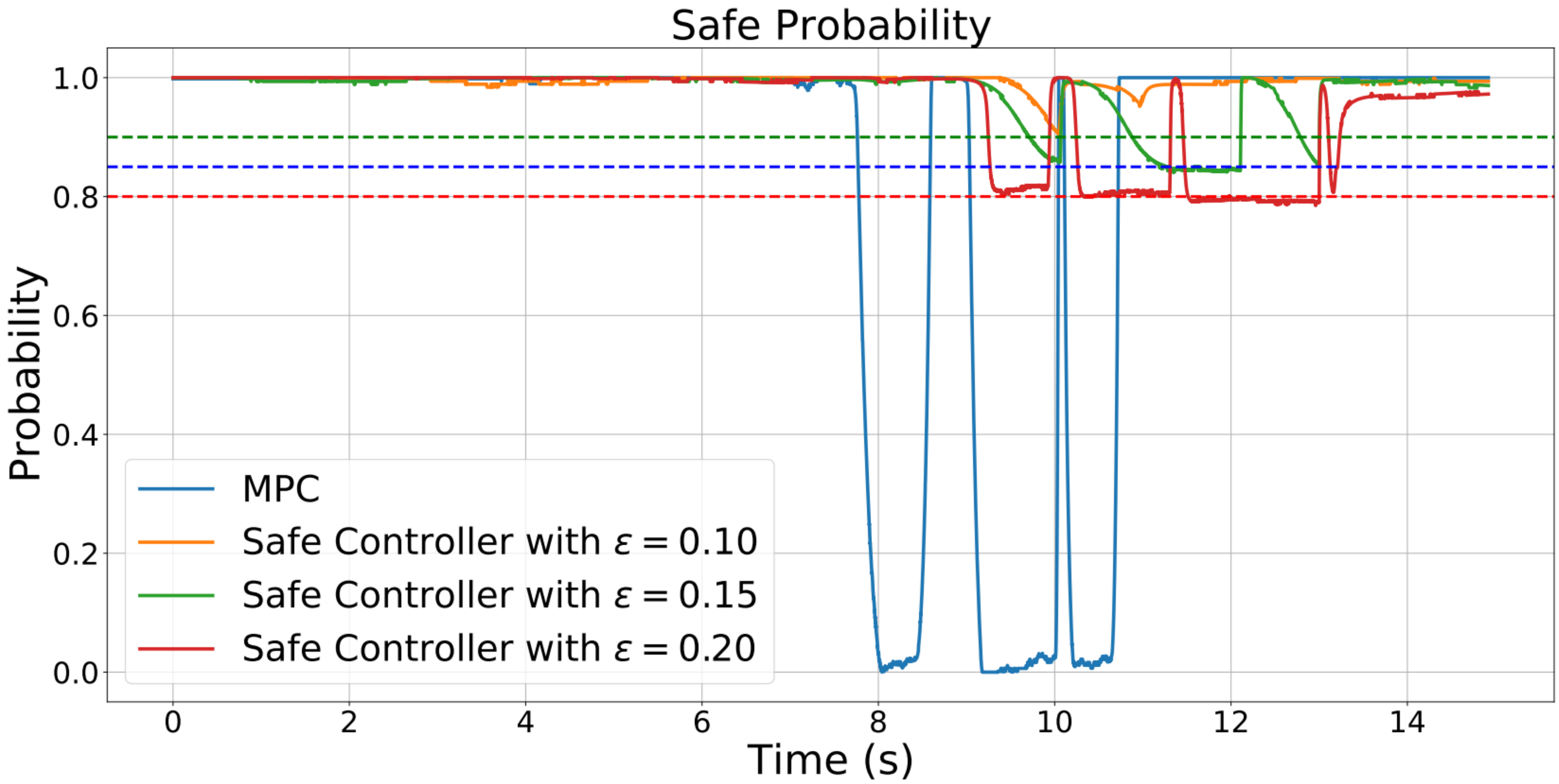
Ahmadi et al.: $\mathrm{CVaR}_\beta(\phi(X_{t+1})) \geq \gamma\phi(X_t)$



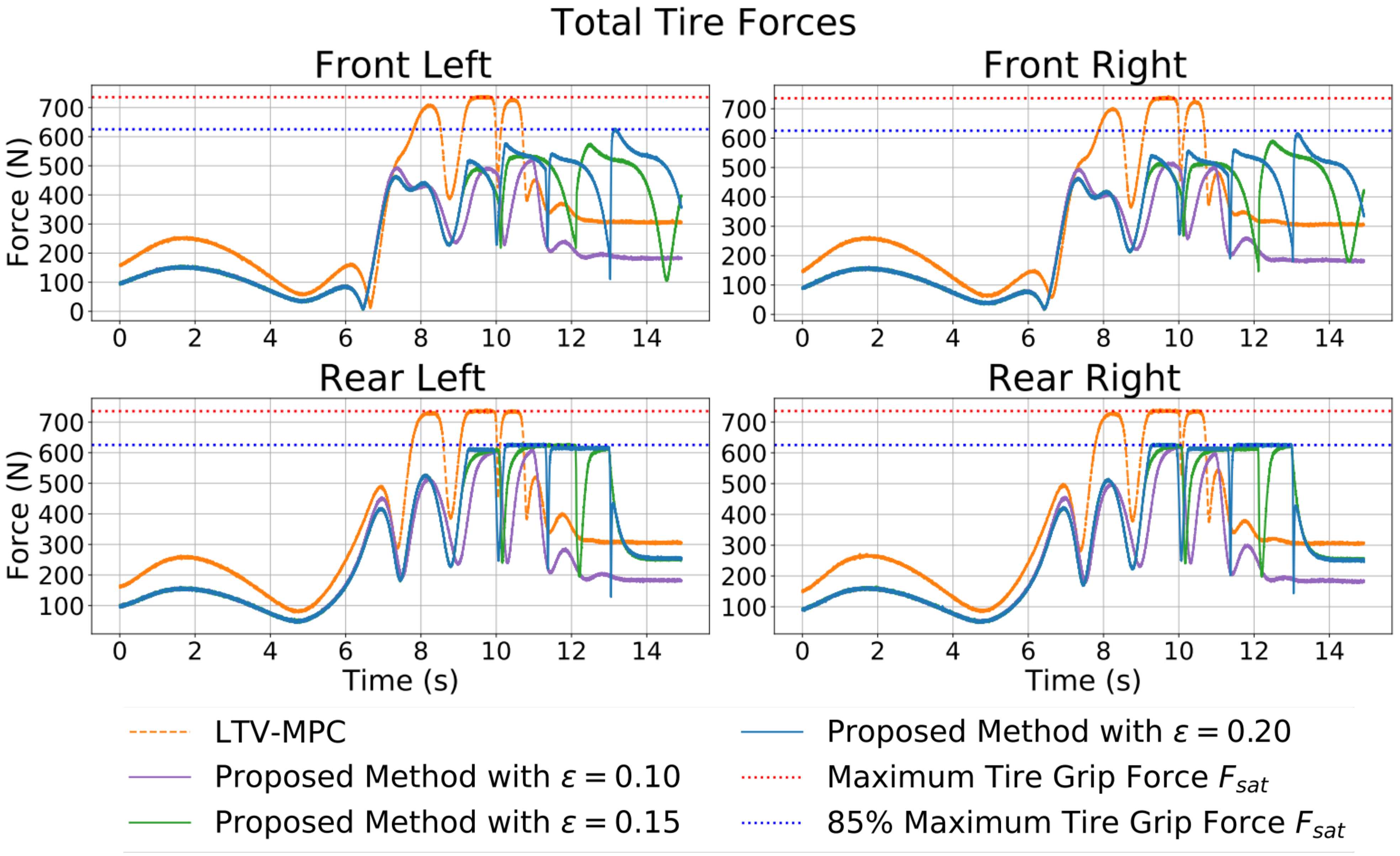
Simulation



Advantage 1: Long-term Safety Guarantee

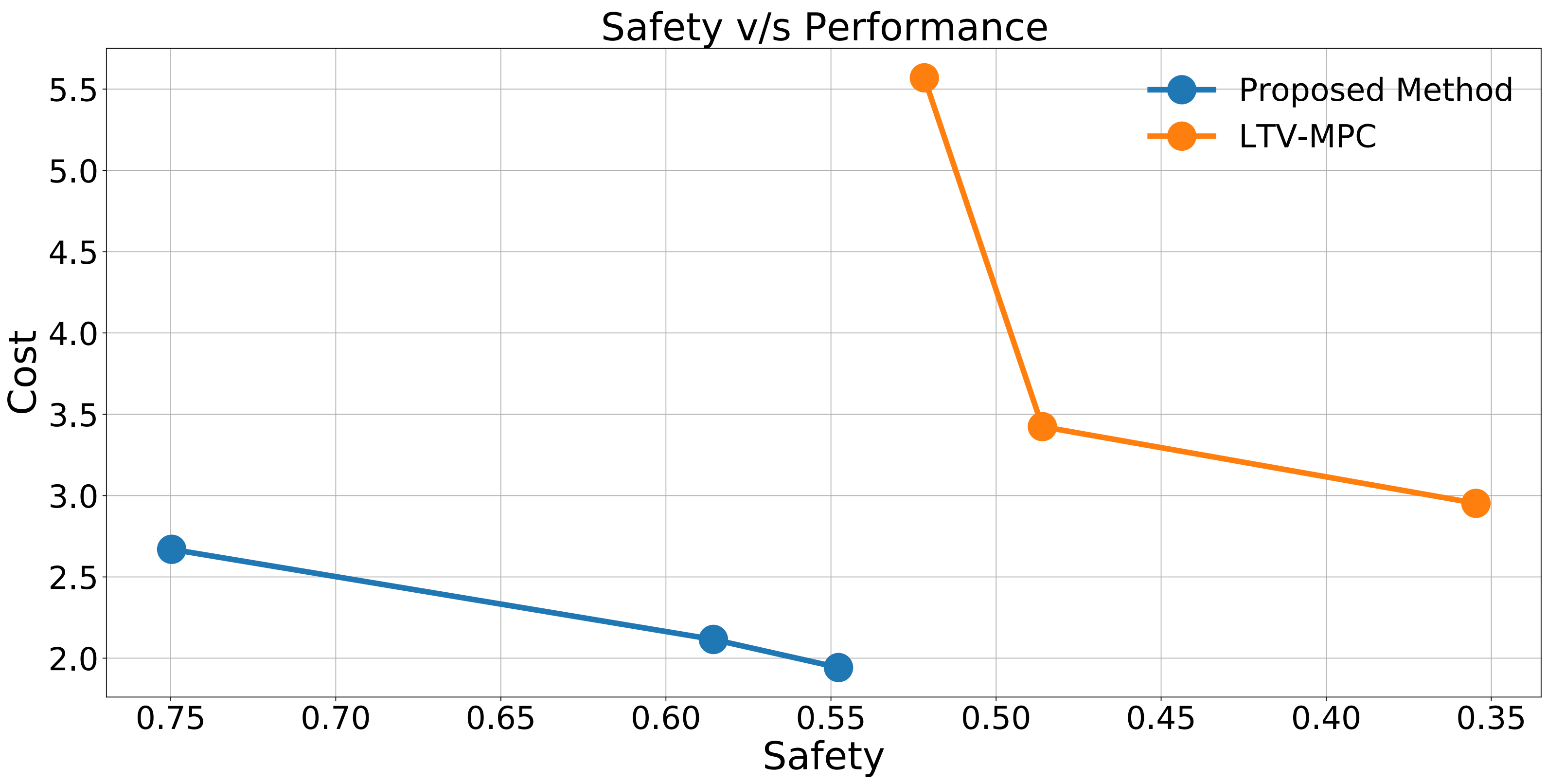


Advantage 1: Long-term Safety Guarantee (Cont'd)



Advantage 2: Better Performance Tradeoffs

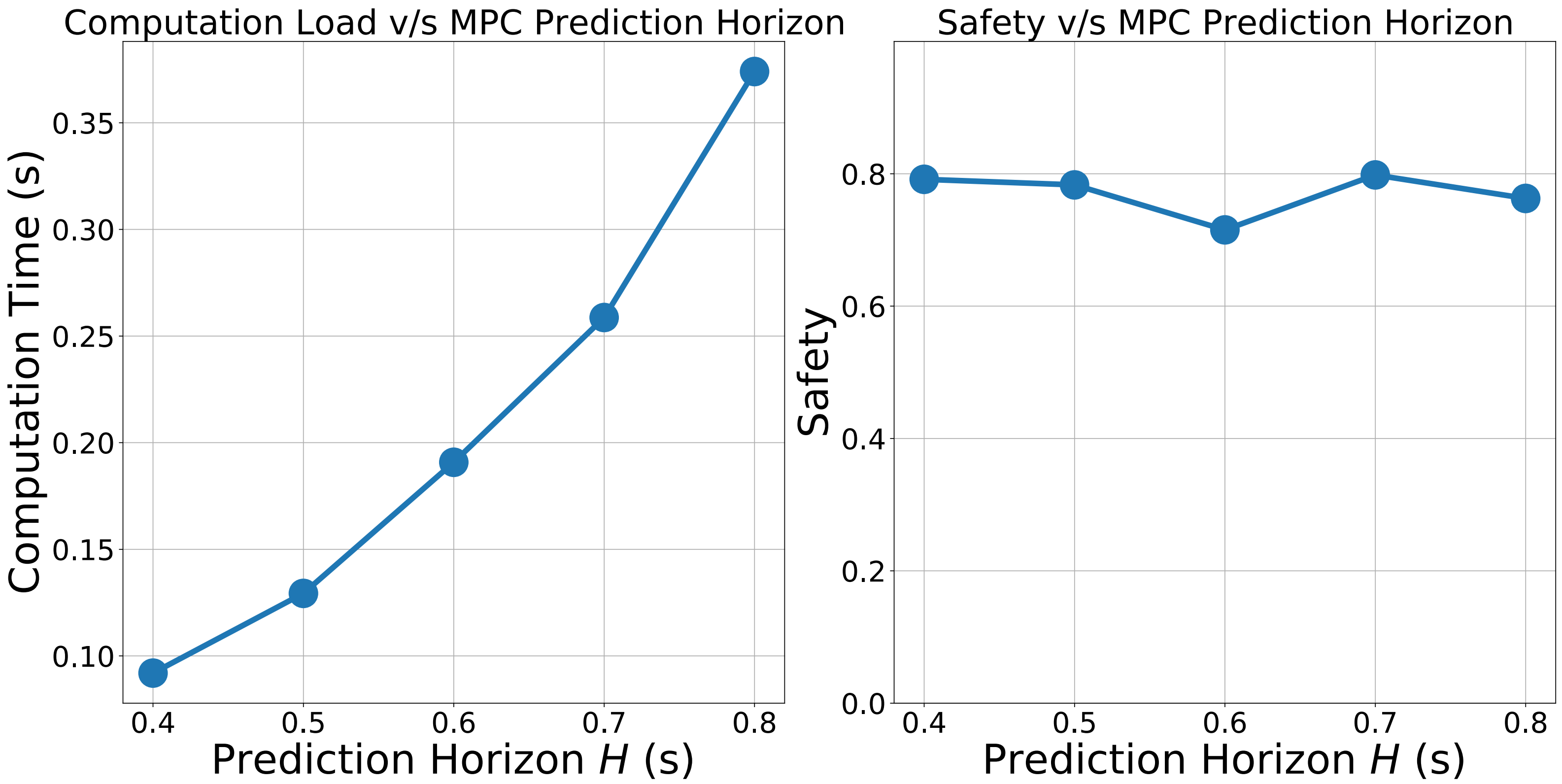
cost:
deviation from the reference trajectory



safety: satisfaction of the tire force limits

Advantage 3: Less Computation Costs

- Computation of MPC grows in $O(H^3)$
- Safety will not be compromised even with short outlook horizons



Random variables that inform safety

Characterized the distribution of:

- **Worst-case margin:** $\Phi_x(T) \coloneqq \inf\{\phi(X_t) \in \mathbb{R}: t \in [0,T], X_0 = x\}$
- **First exit time:** $\Gamma_x(\ell) \coloneqq \inf\{t \in \mathbb{R}_+: \phi(X_t) < \ell, X_0 = x\}$
- **Distance to the safe set:** $\Theta_x(T) \coloneqq \sup\{\phi(X_t) \in \mathbb{R}, : t \in [0,T], X_0 = x\}$
- **Recovery time:** $\Psi_x(\ell) \coloneqq \inf\{t \in \mathbb{R}_+: \phi(X_t) \geq \ell, X_0 = x\}$

All distributions are given by the deterministic convection-diffusion equations

Theorem 1: Worst-case margin $\Phi_x(T)$

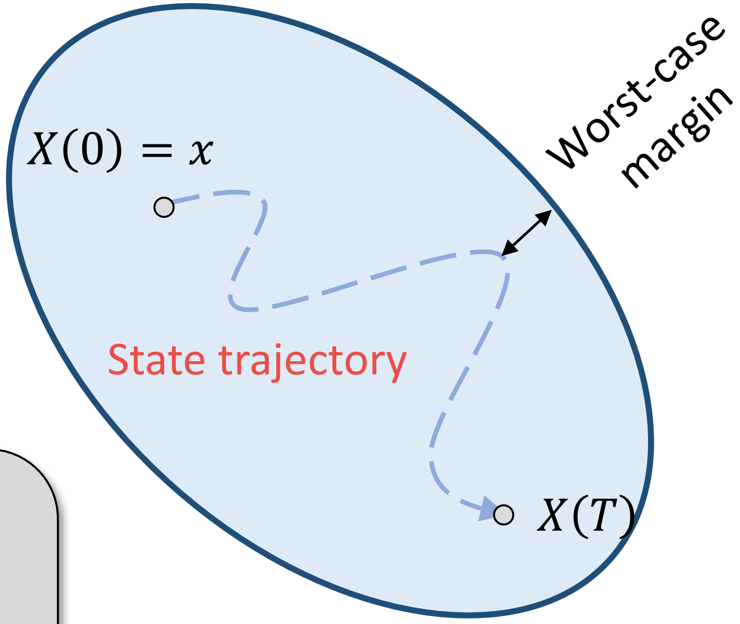
The complementary cumulative distribution function of the safety margin $\Phi_x(T)$

$$F(z,T;\ell) = P(\Phi_x(T) \geq \ell), \ell \in \mathbb{R}$$

is the solution to

$$\begin{cases} \dfrac{\partial F}{\partial t} = \dfrac{1}{2}\nabla\cdot(D\nabla F) + \mathcal{L}_{\rho-\frac{1}{2}\nabla\cdot D}F & z[1] \geq \ell, T > 0 \\ F(z,t) = 1 & z[1] \geq \ell, T > 0 \\ F(z,0) = \mathbf{1}_{\{z[1]<\ell\}}(z) & z \in \mathbb{R}^{n+1} \end{cases}$$

Safe set $\mathcal{C} = \{x: \phi(x) \geq \ell\}$



$F(z,T;0)$ = safe probability during [0,T]
$z = [\phi(x), x]$

Theorem 2: First exit time $\Gamma_x(\ell)$

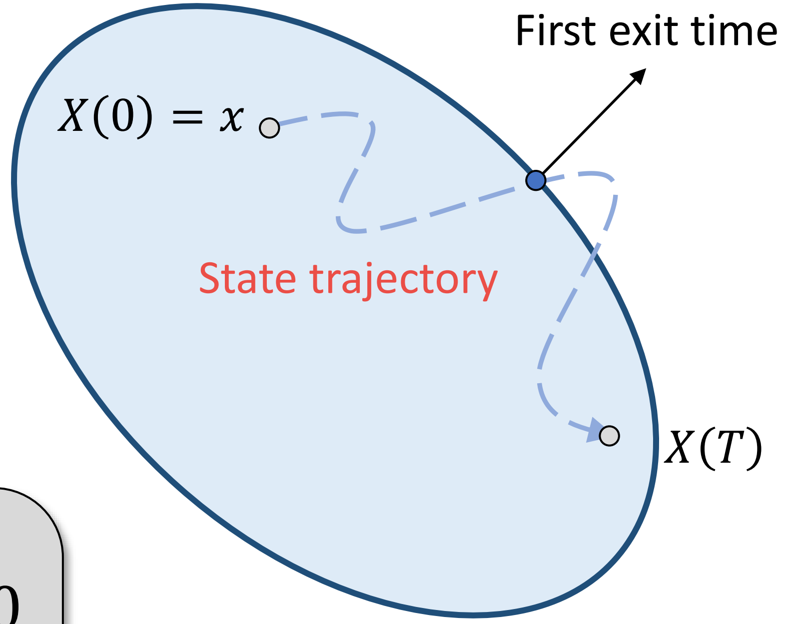
The cumulative distribution function of the first exit time $\Gamma_x(\ell)$

$$G(z,t;\ell) = P(\Gamma_x(\ell) \le t)$$

is the solution to

$$\begin{cases} \dfrac{\partial G}{\partial t} = \dfrac{1}{2}\nabla\cdot(D\nabla G) + \mathcal{L}_{\rho-\frac{1}{2}\nabla\cdot D} G & z[1] \ge \ell, t > 0 \\ G(z,t) = 1 & z[1] < \ell, t > 0 \\ G(z,0) = \mathbb{1}_{\{z[1]<\ell\}}(z) & z \in \mathbb{R}^{n+1} \end{cases}$$

Safe set $\mathcal{C} = \{x : \phi(x) \ge \ell\}$



$1 - G(z,T;0)$ = safe probability during [0,T]

$z = [\phi(x), x]$

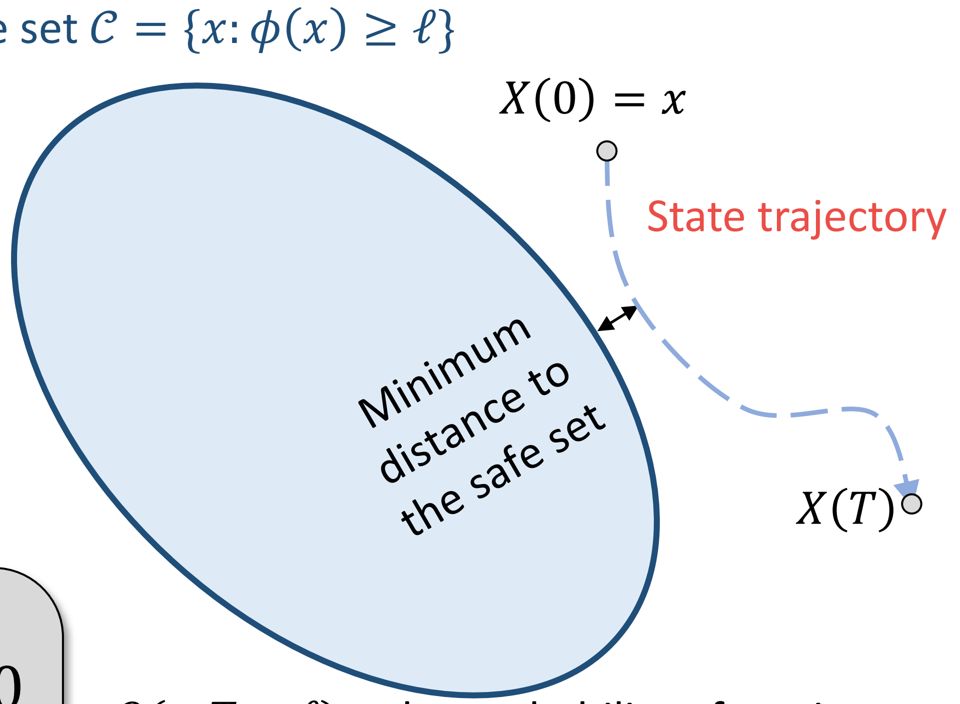
Theorem 3: Distance to the safe set $\Theta_x(T)$

The **cumulative distribution function of the** distance to the safe set $\Theta_x(T)$

$$Q(z,T;-\ell) = P(\Theta_x(T) \geq \ell), \ell \in \mathbb{R}$$

is the solution to

$$\begin{cases} \dfrac{\partial Q}{\partial t} = \dfrac{1}{2}\nabla \cdot (D\nabla Q) + \mathcal{L}_{\rho-\frac{1}{2}\nabla\cdot D} Q & z[1] < -\ell, T > 0 \\ Q(z,T;-\ell) = 0 & z[1] \geq -\ell, T > 0 \\ Q(z,0;-\ell) = \mathbb{1}_{\{z[1]<\ell\}}(z) & z \in \mathbb{R}^{n+1} \end{cases}$$



$Q(z,T;-\ell)$ = the probability of getting within ℓ distance to C during [0,T]

$z = [\phi(x), x]$

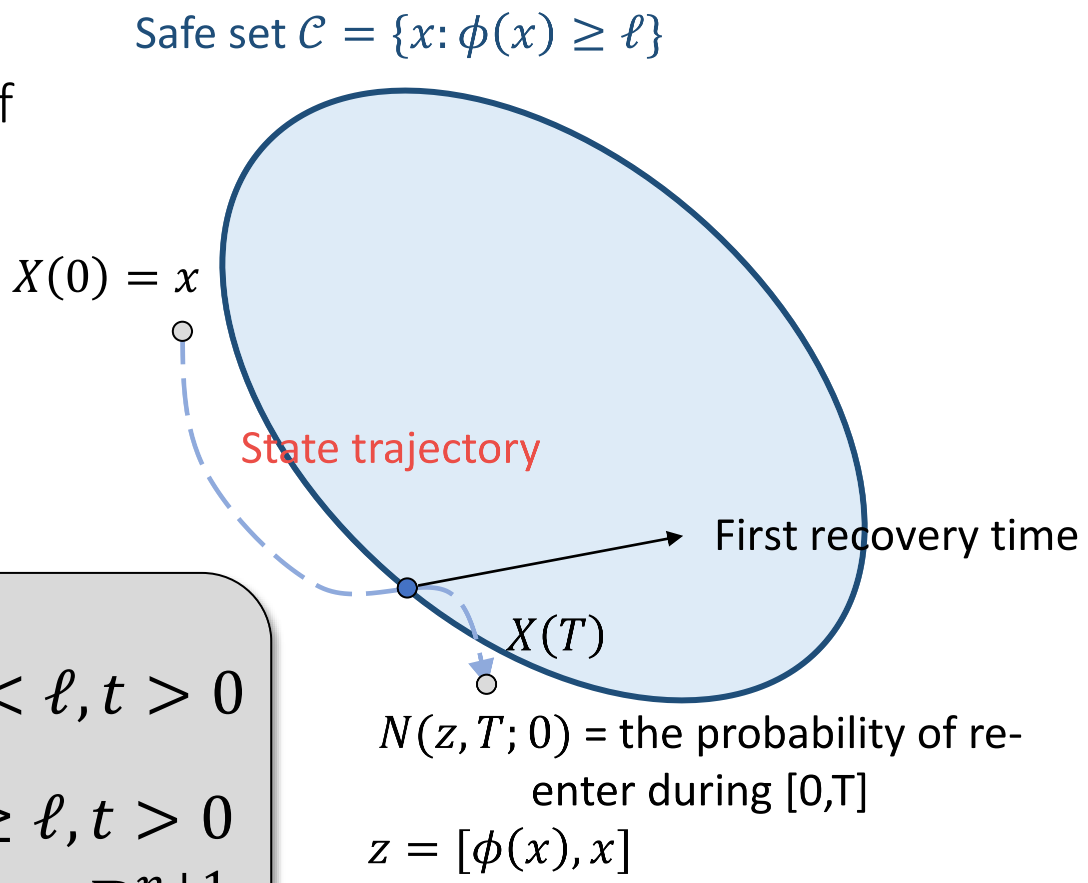
Theorem 4: First recovery time $\Psi_x(\ell)$

Let $D = \zeta\zeta^T$,the cumulative distribution function of recovery time $\Psi_x(\ell)$

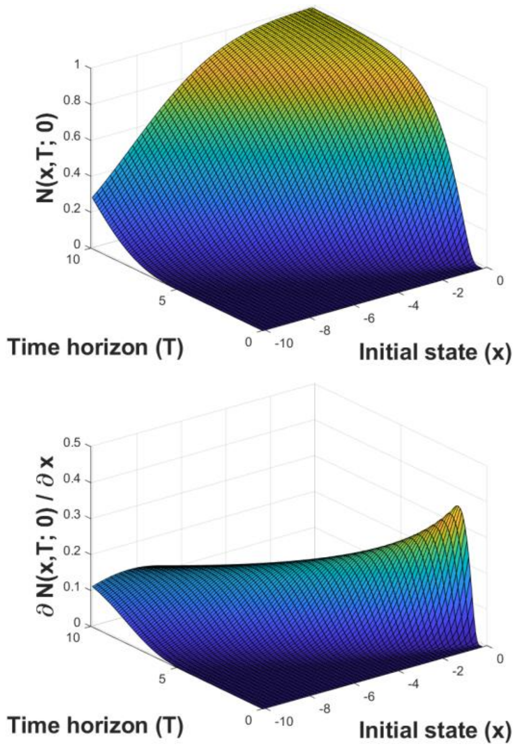
$$N(z,T;\ell) = P(\Psi_x(\ell) \le t)$$

is the solution to

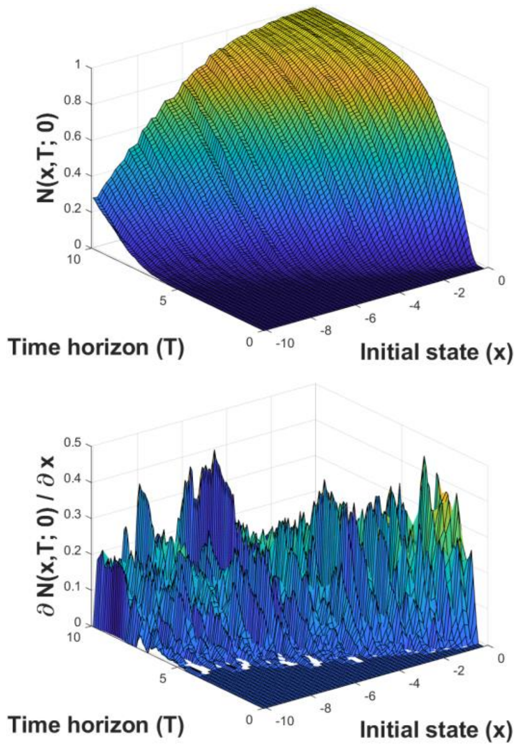
$$\begin{cases} \dfrac{\partial N}{\partial t} = \dfrac{1}{2}\nabla \cdot (D\nabla N) + \mathcal{L}_{\rho - \frac{1}{2}\nabla\cdot D} N & z[1] < \ell, t > 0 \\ N(z,t) = 1 & z[1] \ge \ell, t > 0 \\ N(z,0) = \mathbb{1}_{\{z[1]\ge\ell\}}(z) & z \in \mathbb{R}^{n+1} \end{cases}$$



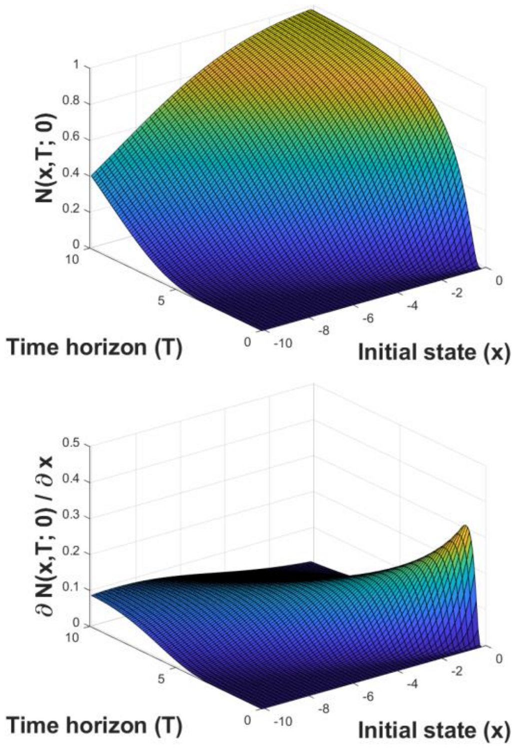
Example use case



Ground truth



Monte Carlo



PDE solver

Today's talk

理学

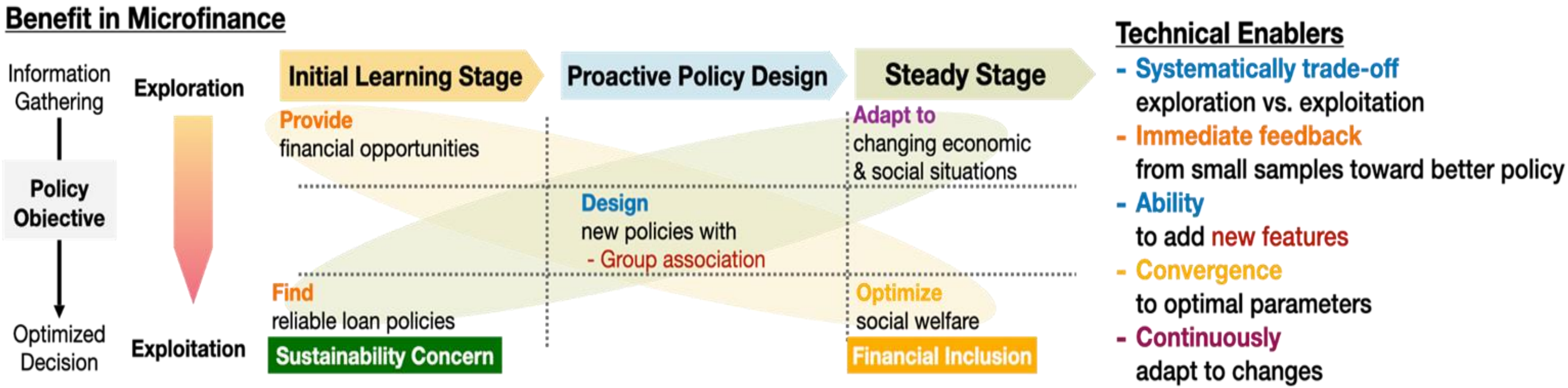
Stochastic safe control
Robust control
Optimization
Information theory …

工学

Microfinance from a control perspective

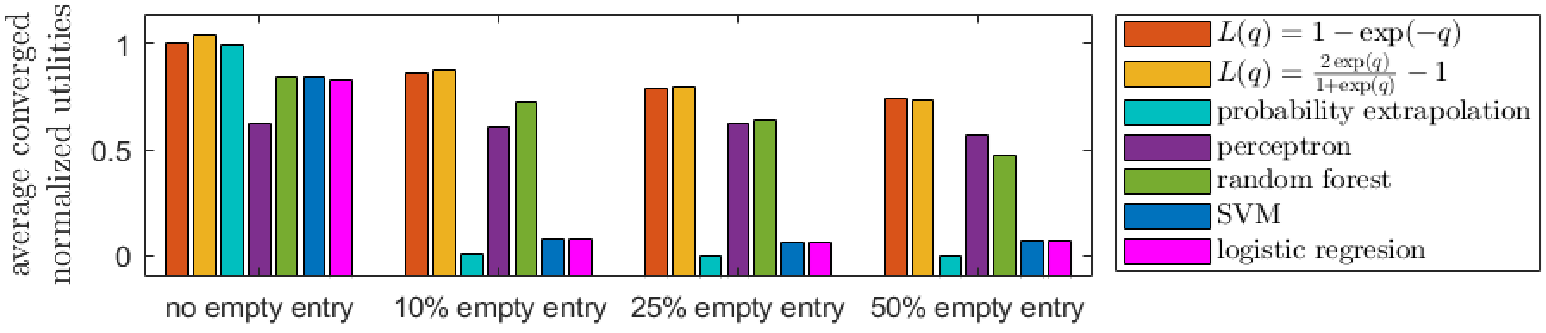
Challenges in microfinance:

1. Complexity in understanding default process
2. Asymmetry, heterogeneity, and incomplete information of individual applications
3. The scarcity of available past data
4. The dynamically evolving social and economic conditions

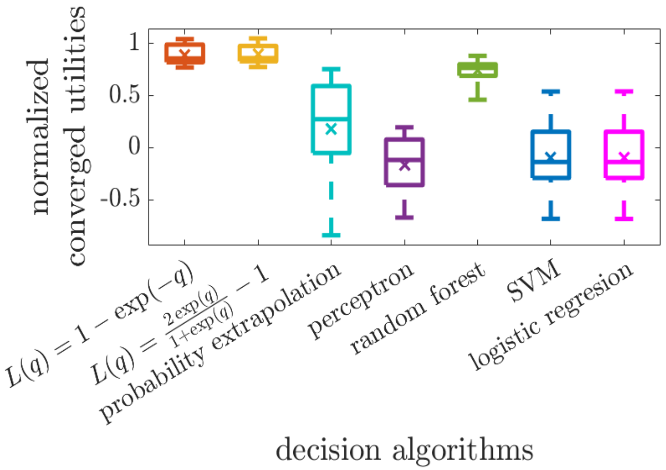


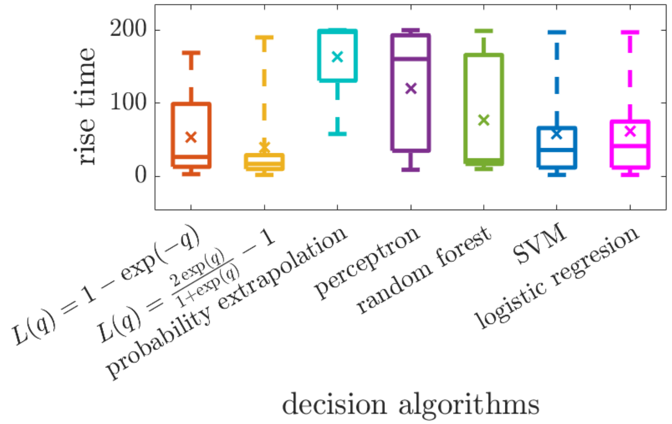
Microfinance from a control perspective

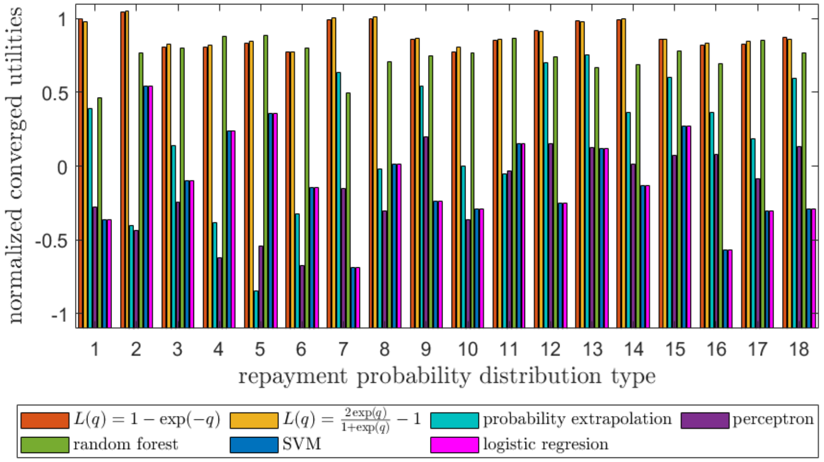
1. **Robustness against missing data**



2. **Ability to deal with diverse microfinance distributions**

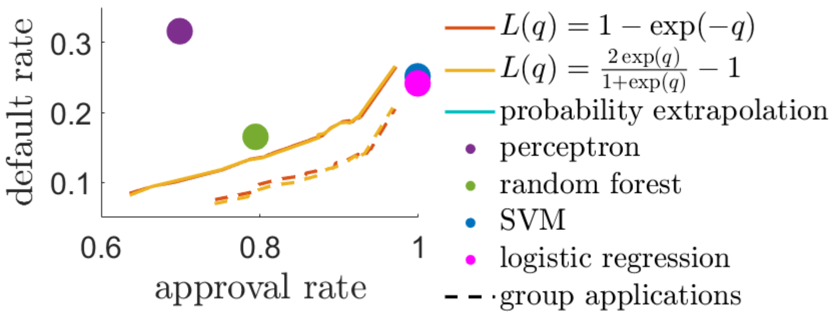




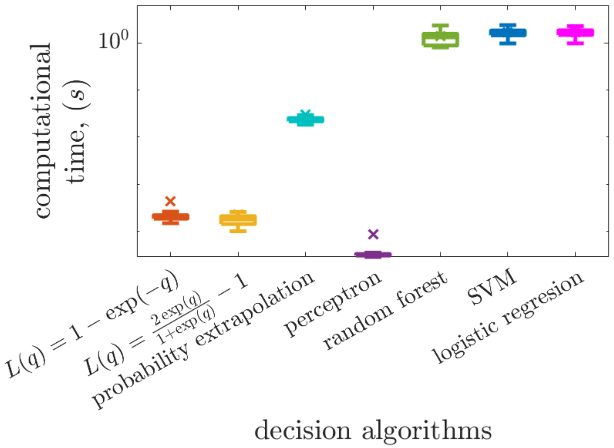


Microfinance from a control perspective

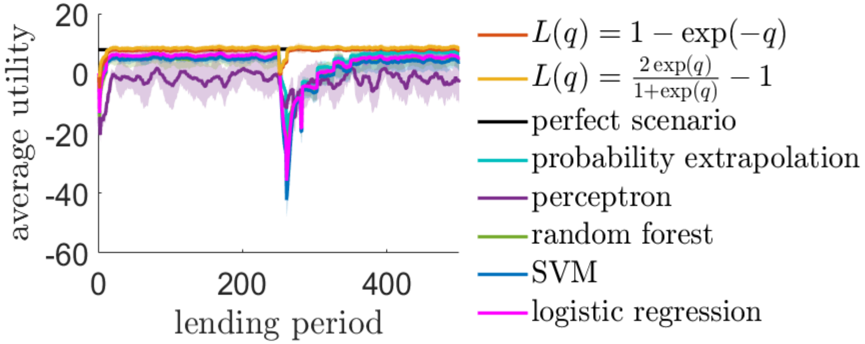
3. **Tradeoff between default rate vs. approval rate**

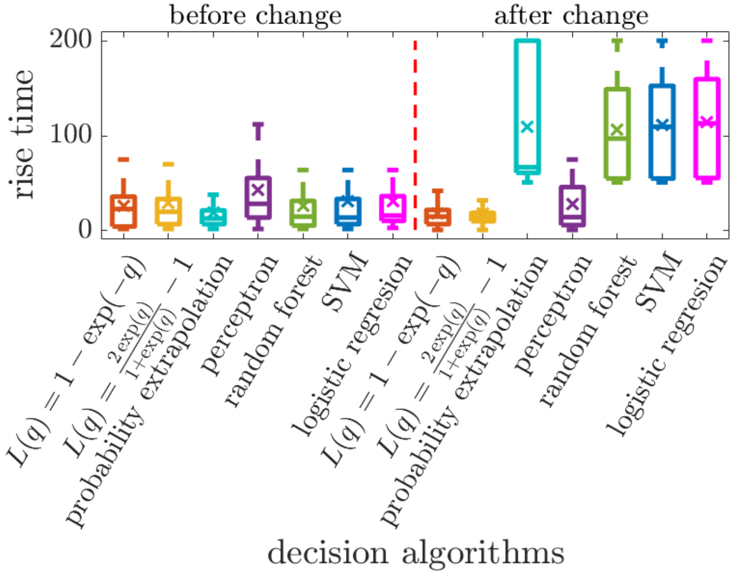


4. **Cheaper computational cost**



5. **Adaptation to changes**





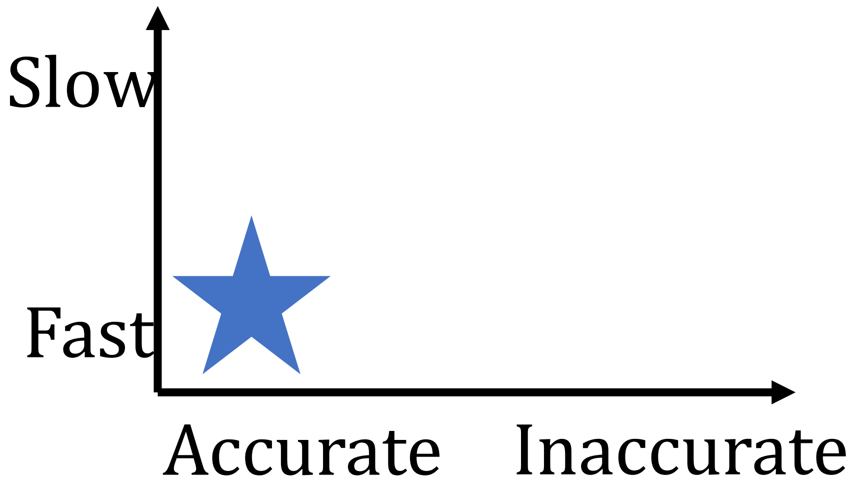
Today's talk

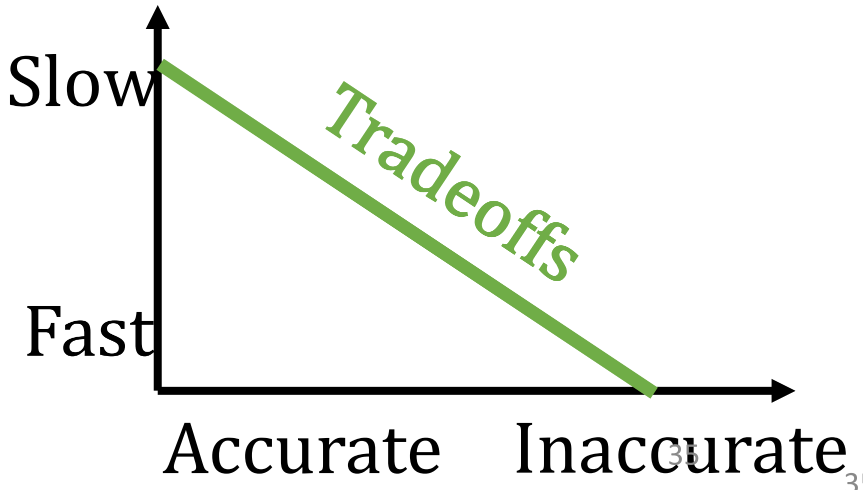
科学

Neuroscience
Biomolecular control…

工学

Constraints vs robust performance in human

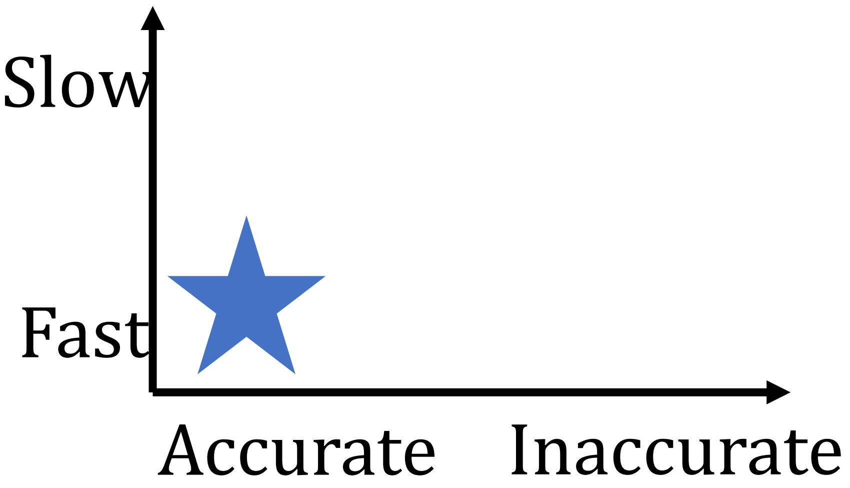




Constraints vs robust performance in human

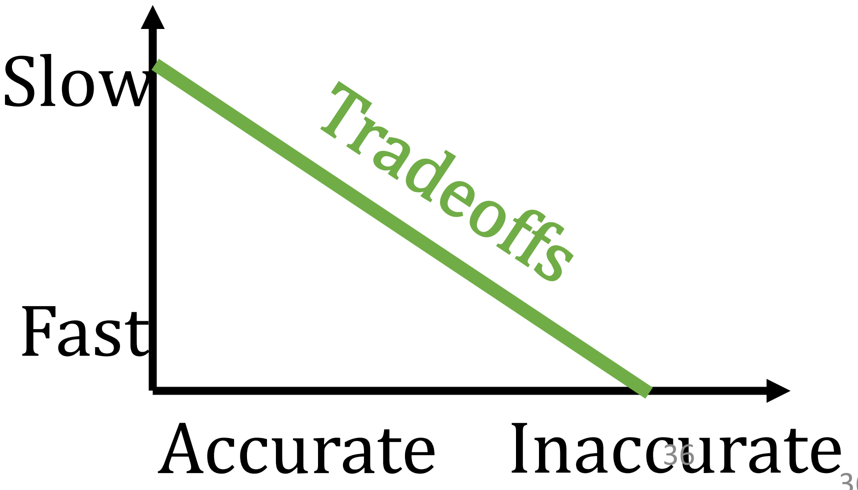
Task 1: Compensate for the head motion

Tokyo Tech



Task 2: Tracking a moving object

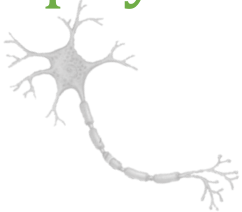
Tokyo Tech



Constraints vs robust performance in human

Sensorimotor control

Neurophysiology



Biking,
eye movement, etc.

↑

A feedback loop
(e.g. VOR, reflex)

↑

Hardware
(neurons, muscles)

↑

Biological resources