

# Strategic Timing and Pricing in On-demand Platforms

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We design a dynamic pricing and allocation mechanism to optimize service provision of an on-demand platform under demand stochasticity, heterogeneity across price-sensitive and time-sensitive customers, and information asymmetry. In this context, *timing* is used as a strategic device to: (i) dynamically manage the imbalances between demand and capacity; and (ii) provide discriminated service levels over heterogeneous customers. The platform always prioritizes service to time-sensitive agents, and extracts all the surplus from price-sensitive agents. Under strong customer heterogeneity, the optimal mechanism involves an extreme form of discrimination by *strategically delaying* all requests from price-sensitive customers, to maximize the price charged to time-sensitive customers. This comes at a loss in total surplus, but the platform extracts all the surplus generated. When customer heterogeneity is weaker, timing is used strategically for both capacity management and discrimination. If the price-sensitive agents are relatively insensitive to wait times, then the platform prioritizes the provision of late services over timely services for price-sensitive customers. Otherwise, demand for late services is no longer prioritized. In this case, no capacity is left strategically idle and the optimal mechanism can even maximize social welfare, but the platform leaves some information rent to the customers. As compared to standard dynamic pricing policies that do not elicit customer preferences, the optimal mechanism can increase platform profits significantly, and may provide a Pareto improvement. Surprisingly, the price charged to time-sensitive agents is *not* an increasing function of realized demand in that a high demand realization might trigger a lower price.

*Key words:* Dynamic Mechanism Design, On-Demand Platforms, Strategic Timing, Dynamic Pricing.

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## 1. Introduction

On-demand platforms have grown to comprise a prevalent part of the modern economy by leveraging independent sellers to serve on-demand requests from potential buyers through dynamic matching. Such platforms are commonly found in transportation (e.g., Uber and Lyft), services (e.g., TaskRabbit), and many other industries. These platforms enable new types of transactions that provide important opportunities to enhance the economic and operational performance of underlying markets.

Several critical features that distinguish on-demand platforms from traditional markets are real-time management of demand and supply, matching capabilities, and flexible and personalized payment schemes. First, imbalances between demand and supply are typically managed in real-time in on-demand platforms in contrast to the traditional approaches to job scheduling and

revenue management. Second, the management of demand and supply can be achieved through matching between buyers and sellers, in contrast to first-come first-served operating procedures. Third, online payment capabilities permit the implementation of real-time personalized pricing schemes. For instance, surge pricing in ride-sharing applies differentiated prices based on spatial-temporal characteristics of rider requests and driver supply, which would be difficult to implement in a physical retail setting.

These platform characteristics enable the design and implementation of novel solutions to the long-standing issues of dynamically managing imbalances between demand and supply in the presence of customer heterogeneity and information asymmetries. From an operational standpoint, traditional approaches based on dynamic pricing and revenue management rely exclusively on public information and may therefore result in lost revenue opportunities and mismatches between service offers and customers' expectations. From an economic standpoint, mechanisms designed to elicit agents' preferences and customize service offerings typically do not focus on real-time demand-supply matching in the face of demand variability. In this paper, we bridge the gap between these two disparate literatures to optimize the dynamic management of demand and capacity while improving the discriminatory capabilities of on-demand platforms.

This paper proposes an original dynamic pricing and allocation mechanism in on-demand platforms in the face of demand stochasticity, customer heterogeneity and information asymmetries. The mechanism relies on the elicitation of customer preferences and leverages this information to provide personalized pricing and service levels. In this context, *timing* is a strategic device for two reasons. First, timing can manage the stochastic and dynamic imbalances between demand and supply by maximizing capacity utilization over time. Second, timing can be used for discriminatory purposes by deliberately delaying service to the price-sensitive customers to charge higher prices to the time-sensitive customers. Specifically, this paper makes the following contributions.

We formalize the environment of an on-demand platform with dynamic imbalances between demand and capacity and heterogeneous customers. The imbalance is captured by stochastic realizations of high and low demand. Customer heterogeneity is captured by a mix of time-sensitive customers (characterized by a high willingness to pay and a low willingness to wait) and price-sensitive customers (characterized by a low willingness to pay and a high willingness to wait). These preferences are private information to the customers. The platform's pricing and allocation problem is formulated in a discrete time setting as an infinite-horizon dynamic program. In each period, the platform elicits customer preferences, and optimizes the price of service and the allocation of capacity to (i) the time-sensitive customers, (ii) the price-sensitive customers who just placed a request, and (iii) the price-sensitive customers who are waiting for late service from earlier

time periods. The model maximizes the platform's expected profit, subject to incentive compatibility, individual rationality and capacity constraints. We first show that the platform extracts all the surplus from the price-sensitive customers, and always prioritizes service provision to all time-sensitive customers. The critical decisions then involve allocating capacity to price-sensitive customers, and determining the price level for time-sensitive customers. We derive a closed-form characterization of the optimal policy.

We identify the structure of the optimal mechanism based on the heterogeneity across time-sensitive and price-sensitive customers and on the time preferences of the price-sensitive customers. First, under strong heterogeneity, the mechanism adopts an extreme form of discrimination by *strategically delaying* all requests from price-sensitive customers. Some capacity is strategically left idle for discriminatory purposes to maximize the price charged to time-sensitive customers. Under weak heterogeneity, the optimal mechanism depends on the time preferences of price-sensitive customers. If price-sensitive customers are relatively insensitive to wait times, the platform prioritizes the provision of late services over timely services for price-sensitive customers because discrimination remains a strong motivation for the platform. Otherwise, the discrimination incentives are weakened and the optimal mechanism is more amenable to providing timely services to price-sensitive customers. In this case, the timing lever is primarily used as a means of smoothing out the imbalances between demand and supply, and no capacity is left strategically idle.

Surprisingly, the optimal price does *not necessarily increase* with the realized demand. To see the intuition behind this result, consider an instance where, under low demand, the platform delays service to the price-sensitive customers to: (i) charge a higher price to the time-sensitive customers, and (ii) serve more of the demand for late services. Under high demand, however, this strategy would create a longer queue for late services, which the platform may not be able to meet in the next period. Therefore, the platform may instead provide more timely services to the price-sensitive customers under high demand, which implies a lower price charged to the time-sensitive customers to maintain incentive compatibility.

We compare our mechanism with (i) the first-best allocation rule that maximizes social surplus under perfect information, and (ii) a dynamic pricing policy that does not elicit customers' preferences (e.g., surge pricing in the ride-sharing context). First, we show that, under strong customer heterogeneity, the optimal mechanism induces a loss in social surplus as compared to the first-best allocation. In this case, the platform may nonetheless be able to capture all the surplus generated without leaving any information rent to the customers. Vice versa, under weak customer heterogeneity, we identify a regime where the optimal mechanism achieves the first-best allocation. In this case, however, the platform leaves some surplus to the time-sensitive customers as information rent due to the information asymmetries. Second, we show that the optimal mechanism results in

strictly larger platform profits than baseline surge pricing policies, and that the value of information regarding customers' preferences can be significant. Moreover, we identify regimes where it even provides a Pareto improvement across all participants.

These results suggest potential opportunities to enhance the economic and operational performance of on-demand platforms by eliciting customer preferences and adjusting prices and service levels accordingly. The insights from this paper are in line with recent industrial developments such as Uber Pool, Uber Express Pool and Lyft Line. These provide differentiated services that implicitly account for heterogeneity in time preferences. In this paper, we design a mechanism that explicitly achieves similar objectives *without resorting to the development of new products or services*. In fact, Kakao Taxi, the dominant ride-sharing company in Korea, recently launched a new option to enable fast pickup at a price premium. This provides a prime example of the type of time discriminatory mechanism proposed in this paper.

The remainder of this paper is organized as follows. We review the related literature in Section 2. Section 3 develops our pricing and allocation mechanism and formulates it as an infinite-horizon dynamic program. The optimal policy is then characterized in Section 4 over the entire parameter space. It outlines the main drivers of the platform's pricing and allocation policies as a function of valuation heterogeneity, time preferences, and demand patterns. Section 5 compares our proposed mechanism to a first-best allocation rule based on perfect information, and to a baseline mechanism inspired from surge pricing in the ride-sharing context. It quantifies the impact of our mechanism on platform profitability and economic efficiency. Section 6 concludes.

## 2. Related Literature

This paper contributes to the growing literature on the design and optimization of on-demand platforms. Most related to this paper, two problems that have garnered particular attention are matching and pricing.

First, the matching problem involves assigning available sellers to each incoming customer request. This builds upon the theory of matching markets, which has been applied to such problems as kidney exchanges, school assignment and housing markets (Roth et al. 2004, Abdulkadiroğlu et al. 2009, Leshno 2017, Arnosti et al. 2018). In the context of on-demand platforms, Hu and Zhou (2016) design dynamic matching policies for profit maximization in the face of impatient buyers and sellers who may leave the platform if they remain unmatched. In ride-sharing, the problem is complicated due to the spatial dynamics of demand and supply. Ozkan and Ward (2018) propose a linear programming algorithm that leverages demand forecasts, and the uncertainty thereof, in matching arriving riders with available drivers. They show that the widely adopted policy that matches requests to the closest available driver does not necessarily maximize the

number of transactions operated through the platform. In addition, Wang et al. (2017) consider the potential mismatches between system-optimal matches and user-optimal matches, and developed mathematical approaches to generate stable approaches to matching in ride-sharing platforms.

On the pricing side, the revenue management and dynamic pricing literature has studied how firms should dynamically adjust their inventory and pricing levels to match supply and demand in the face of capacity constraints (Talluri and Van Ryzin 2006, Bitran and Caldentey 2003, Talluri et al. 2008, Özer and Phillips 2012). Recent studies have incorporated strategic customers, i.e., customers that can strategically time their purchases for utility maximization (Board 2008, Lobel et al. 2015, Board and Skrzypacz 2016). Closely related to our work, Besbes and Lobel (2015) optimize dynamic pricing policies in a setting with heterogeneous customers that exhibit differentiated willingness to pay and willingness to wait. This policy relies on a posted price mechanism that applies uniform prices across the full customer population at any time of purchase. In contrast, we focus in this paper on a direct mechanism that enables the customers to dynamically report their preferences to the platform, and designs a pricing and allocation policy that leverages this information.

Several recent papers have addressed the problem of dynamic pricing in the context of on-demand platforms, especially in a strategic queuing setting where customers balance price levels and wait times (Banerjee et al. 2015, Taylor 2017, Bai et al. 2018). In ride-sharing, the use of surge pricing has also attracted recent interest. For instance, Cachon et al. (2017) point out the potential benefits of surge pricing, as compared to static pricing practices, in the face of dynamic imbalances between demand and supply. Bimpikis et al. (2016) abstract away from the temporal dynamics of the system, and focus on the spatial ride-sharing pricing problem. They show that a more “balanced” distribution of the demand over the network translates into a greater consumer surplus and platform profit. Guda and Subramanian (2018) analyze the role of surge pricing in managing the availability of the drivers across several locations. They suggest that surge pricing in low-demand locations may be optimal in some circumstances (referred to as “strategic surge pricing”). This result shares some similarities with our insight that the optimal price is not necessarily monotonic with demand, but this comes from a very different rationale. In their setting, strategic surge pricing is used to incentivize drivers to relocate to high-demand locations, while, in this paper, the non-monotonicity of prices stems from the use of strategic timing for dynamic demand-supply management and discrimination in the face of customer heterogeneity.

Our paper also relates to the economic literature on dynamic mechanism design (see Bergemann and Said (2011) for a good survey). This literature studies a class of problems in which customers arrive dynamically and stochastically onto a market and have private information regarding their service preferences, their willingness to pay, and their time sensitivities. The principal aims to

elicit information and design corresponding pricing and allocation policies for profit maximization (Battaglini 2005, Said 2012, Pai and Vohra 2013, Kakade et al. 2013). One of the main distinctions of our framework from these papers is that, motivated by the context of on-demand platforms, the seller’s capacity is perishable, i.e., cannot be transferred from one period to the next. This strengthens the potential benefits from the use of the timing lever for pricing and allocation.

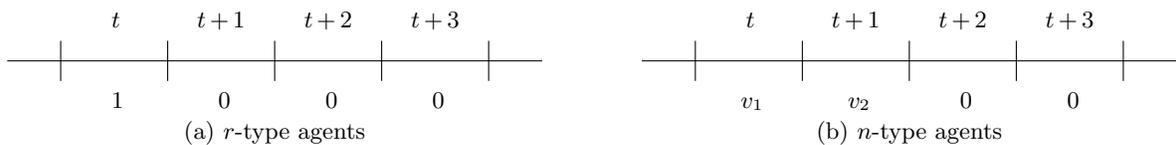
### 3. Model

We first develop the model’s settings and our main assumptions. We then formulate the platform’s optimization problem, and derive initial insights to simplify the formulation.

#### 3.1. Setting and Assumptions

We consider an on-demand platform that operates continuously over time, and matches suppliers with a demand of agents<sup>1</sup>. We consider a setting with heterogeneous agent preferences that are private information, and stochastic demand.

**Agent preferences:** On the demand side, we assume that there are two types of agents: (i) *time-sensitive* agents (rush type, referred to as *r-type* henceforth), and (ii) *price-sensitive* agents (non-rush type, referred to as *n-type* henceforth). A time-sensitive agent only values timely service, and is willing to pay a price premium to receive timely service. A price-sensitive agent, on the other hand, is not willing to pay as much for any service, but would accept a delayed service from the platform. Specifically, an *r-type* agent receives a positive utility only if the service is provided when he arrives into the platform. This utility value is normalized to 1 without loss of generality. An *n-type* agent, on the other hand, receives a positive utility from a *timely service* (i.e., a service assigned at the arrival period) as well as from a *late service* (i.e., a service provided in the subsequent period). The corresponding values are equal to  $v_1$ , and  $v_2$  respectively. We assume that  $1 > v_1 > v_2 \geq 0$ , i.e., *n-type* agents derive a lower value than *r-type* agents from timely services, reflected in the difference 1 and  $v_1$ , and incur a cost of waiting, reflected in the difference between  $v_1$  and  $v_2$ . Note that under this formulation, there is perfect correlation between time preferences and willingness to pay for a timely service. This is summarized in Figure 1.



**Figure 1** Time-dependent utility of *r-type* and *n-type* agents joining the platform in period  $t$

<sup>1</sup>Throughout this paper, we refer to the buyers, or customers as “agents”.

Agent types are identically and independently distributed over time. We denote by  $\sigma$  the probability that an incoming agent is of  $r$ -type. The value of  $\sigma$  is commonly known, but each agent type is private information. This creates information asymmetry between the platform and the agents, and motivates the design of an incentive compatible mechanism to elicit this information and leverage it in its pricing and allocation decisions. Each agent aims to maximize his expected utility, which is equal to the expected value of the service that he receives from the platform (if any), minus the expected payment.

**Demand-supply imbalances:** Demand and supply feature stochastic imbalances over time. We consider a continuum of suppliers, which stays constant over time and which we normalize to unit mass without loss of generality. At each time period, a continuum of agents of mass  $D$  (demand) request a service on the platform. We assume that the demand can be either high ( $D = H$ ) or low ( $D = L$ ), with  $0 < L < 1 < H$ . Therefore, the platform faces an excess of supply in low-demand periods and a shortage of supply in high-demand periods. The demand realizations are independent and identically distributed over time, and we denote by  $k$  the probability of high demand. Finally, we assume that every service takes exactly one period to be completed, and all the suppliers are ready to serve another agent in the next time period.

**Platform problem:** At each time period, the platform optimizes the prices charged to the agents and the allocation of suppliers to serve each agent request. Its objective is to maximize expected discounted profits over an infinite horizon. We formalize this problem in a discrete time dynamic setting with a discount rate of  $\delta < 1$ .

To focus on pricing and allocation, we abstract away from the supply side incentives by assuming that the platform collects all the amount received from the buyers. This abstraction is equivalent to a setting in which the suppliers are homogeneous and have a constant outside option (i.e., the opportunity cost of providing service within the platform) normalized to 0. As a result, the demand side is the only source of information asymmetry, and the platform accrues all the amount collected from the agents without leaving any positive surplus to the suppliers.

From the revelation principle, we restrict our attention without loss of generality on the set of direct mechanisms in which the agents report their types upon arriving to the platform.<sup>2</sup> The mechanism then specifies a contingent *allocation rule* that specifies a probabilistic service provision, and a corresponding *payment rule* for each agent type. We assume that the platform does not discriminate over the agents of the same type, i.e., all the agents who report the same type

<sup>2</sup> Note that we assume that agents request service and report their types at their time of arrival. This differs from the setting of Besbes and Lobel (2015), which focuses on agents' strategic timing of purchase. However, as we shall see, this is without loss of generality in our setting because agents would not have any incentive to strategically delay their entry under the optimal mechanism.

are treated equally. In the most general mechanism of this form, the platform could ask the agents to pay a certain amount conditional on the reported type regardless of the realized assignments. However, in most real-life applications, each agent has the option to opt out from the transaction after placing a request. This option translates into an ex-post individual rationality constraint that the mechanism has to satisfy, i.e., each agent must receive a non-negative payoff after each realization of the stochastic allocation rule. But then any mechanism satisfying this ex-post individual rationality constraint can be implemented by specifying a price for timely and late services that each agent pays only if the corresponding service is provided. Therefore, we restrict our attention on the mechanisms with a payment rule defined on a per-service basis.

The platform’s decisions fall into three categories. First, each agent that reports an  $r$ -type stays within the platform for at most for one period. Therefore, the mechanism specifies the probability that a timely service will be provided, denoted by  $q_r$ , and a per-service price, denoted by  $p_r$ . An  $n$ -type agent, on the other hand, stays in the platform for an additional period in case he is not provided a timely service. Thus, for an agent that reports his type as  $n$ , the mechanism specifies (i) the probability of getting a timely service, denoted by  $q_t$ , and the price of a timely service, denoted by  $p_t$ , as well as (ii) the probability of getting a late service *conditionally* on not getting a timely service in the period of arrival, denoted by  $q_l$ , and the price of a late service, denoted by  $p_l$ .

The design of the pricing and allocation mechanism considered here gives rise to the following trade-offs. First, when the realized demand is high, the platform faces excess demand. Therefore, the platform will aim to prioritize the allocation of its suppliers to the  $r$ -type agents, at a price premium, and to transfer some  $n$ -type agents to the next period. The platform can then provide a late service or reject the request altogether, depending on the realized demand in the next period. When the realized demand is low, the platform might be able to provide a timely service to all the agents, but this might not be optimal if the number of agents waiting for a late service from the previous period is large enough. In either case, the platform may still prefer to transfer  $n$ -type agents to the subsequent period for discriminatory purposes, i.e., to charge a higher price to the time-sensitive agents while ensuring incentive compatibility. We formalize these trade-offs and formulate the resulting profit maximization problem from the platform’s standpoint.

### 3.2. Dynamic Programming Formulation

In the infinite-horizon dynamic program, the state variable includes (i) the realized demand in the period considered  $D$ , and (ii) the number of  $n$ -type agents transferred from the previous period and waiting for a late service (denoted by  $\Gamma$ ). We assume that the state  $(D, \Gamma)$  is publicly observed by the agents upon reporting their types.<sup>3</sup> The mechanism then optimizes the pricing rules (i.e.,  $p_r$ ,  $p_t$  and  $p_l$ ) and the allocation rules (i.e.,  $q_r$ ,  $q_t$  and  $q_l$ ), contingent on the state variable  $(D, \Gamma)$ .

<sup>3</sup> In practice, consumers can observe signals that are proxies for demand realizations (e.g., weather conditions, traffic in the ride-sharing context).

The *transition* of the state variable is as follows. Let  $(D', \Gamma')$  be the state in the upcoming period, which is determined by the state  $(D, \Gamma)$  as well as the allocation rule that the mechanism implements in the current period. First, since the demand realization follows an identical and independent distribution over time regardless on the mechanism considered,  $D'$  is exogenous, and equal to  $H$  or  $L$  with respective probabilities  $k$  and  $1 - k$ . In contrast,  $\Gamma'$  is endogenous and given by  $\Gamma' = (1 - q_t)(1 - \sigma)D$ . This is due to the facts that, in each period, a mass  $(1 - \sigma)D$  of  $n$ -type agents arrive on the platform, a fraction  $q_t$  of them receive a timely service, that the remaining fraction will be transferred to the next period. Moreover, since all  $r$ -type agents stay in the platform for one period only,  $q_r$  does not have any impact on  $\Gamma'$ .

Given the allocation and price decisions, the expected payoffs of  $r$ -type and  $n$ -type agents, contingent on the state  $(D, \Gamma)$ , are denoted by  $U_r$  and  $U_n$ , respectively. For an  $r$ -type agent the expected payoff depends solely on the probability of getting a (timely) service and its price:

$$U_r = q_r(1 - p_r). \quad (1)$$

The payoff of  $n$ -type agents includes the expected utility derived from a timely service, which is assigned with probability  $q_t$  at price  $p_t$ , and the expected utility derived from a late service, which is assigned in the next period with probability  $(1 - q_t)q_l(D', \Gamma')$  at price  $p_r(D', \Gamma')$ .<sup>4</sup> Note that the late service probability is written as the probability of not being assigned a timely service multiplied by the conditional probability of being assigned a late service in the subsequent period. Given the stochasticity of  $D'$ , the expected utility of an  $n$ -type agent in state  $(D, \Gamma)$  is:

$$U_n = q_t(v_1 - p_t) + (1 - q_t)[kq_l(H, \Gamma')(v_2 - p_l(H, \Gamma')) + (1 - k)q_l(L, \Gamma')(v_2 - p_l(L, \Gamma'))]. \quad (2)$$

We now turn to the incentive compatibility constraints for  $r$ -type and  $n$ -type agents, denoted by  $\mathcal{IC}_r$  and  $\mathcal{IC}_n$ , respectively. They ensure that the expected utility of each agent is not less than the one that arises from misreporting his type. If an  $r$ -type agent misreports his type as  $n$ , his payoff is equal to the value derived from a timely service, i.e.,  $1 - p_t$ , multiplied by the probability of getting a timely service, i.e.,  $q_t$ . The allocation of late services is irrelevant, since  $r$ -type agents leave the platform when not provided a timely service. If an  $n$ -type agent misreports his type as  $r$ , he will not be provided a late service by the platform, so his resulting expected utility is equal to the value derived from a timely service, i.e.,  $v_1 - p_r$ , multiplied by the probability of getting a timely service, i.e.,  $q_r$ . Therefore, the constraints are expressed as follows:

$$\mathcal{IC}_r: \quad U_r \geq q_t[1 - p_t]. \quad (3)$$

<sup>4</sup> In order to distinguish the late services provided in the current period and the subsequent one, we use here  $q_l(D', \Gamma')$  and  $p_l(D', \Gamma')$  to refer to the values of the variables  $q_l$  and  $p_l$  in the next period contingent on the state variable  $(D', \Gamma')$ . This slight abuse of notation makes the exposition clearer, without impacting the further developments.

$$\mathcal{IC}_n: \quad U_n \geq q_r[v_1 - p_r]. \quad (4)$$

Throughout this paper, we assume that serving all  $r$ -type agents is always feasible regardless of the realized demand. In other words, even under high demand, the total mass of  $r$ -type agents is lower than the unit mass of suppliers.<sup>5</sup> Mathematically, it translates into the following condition.

ASSUMPTION 1.  $\sigma H < 1$ .

We now formulate the platform's optimization problem, which we denote by  $(\mathcal{P})$ . Let  $V(D, \Gamma)$  be the value function of the problem. The Bellman equation is then given by:

$$V(D, \Gamma) = \max_{\substack{q_r, q_t, q_l \\ p_r, p_t, p_l}} p_r q_r \sigma D + p_t q_t (1 - \sigma) D + p_l q_l \Gamma + \delta [kV(H, \Gamma') + (1 - k)V(L, \Gamma')] \quad (5)$$

$$\text{s.t. } \mathcal{IC}_r \text{ and } \mathcal{IC}_n, \quad (6)$$

$$p_r \leq 1, \quad p_t \leq v_1, \quad p_l \leq v_2, \quad (7)$$

$$1 \geq q_r \sigma D + q_t (1 - \sigma) D + q_l \Gamma, \quad (8)$$

$$\Gamma' = (1 - q_t)(1 - \sigma) D. \quad (9)$$

Equation (5) maximizes the platform's expected profit in the current period and its discounted future expected value over the next demand realization. The term  $p_r q_r \sigma D$  corresponds to the expected revenue from timely services to  $r$ -type agents, equal to their price  $p_r$  multiplied by their probability  $q_r$  and the mass of  $r$ -type agents  $\sigma D$ . Similarly,  $p_t q_t (1 - \sigma) D$  and  $p_l q_l \Gamma$  correspond to the expected revenues derived from timely and late services provided to  $n$ -type agents, respectively. Equation (6) includes the incentive compatibility constraints. Constraint (7) expresses individual rationality constraints to make sure that the platform never charges more for a service than the agents' valuations, and thus guarantees the *ex post* participation of the agents. Constraint (8) is the resource constraint, which imposes that the total number of services cannot be larger than the number of suppliers present in the platform (normalized to 1). Finally, Constraint (9) defines the transition of the system from one period to the next. In the remainder of this section, we denote the decision variables  $(q_r, q_t, q_l, p_r, p_t, p_l)$  as a function of the state variable  $(D, \Gamma)$ . Proposition 1 shows that  $(\mathcal{P})$  admits a solution.

PROPOSITION 1. *There exists a solution to problem  $(\mathcal{P})$ .*

<sup>5</sup> Otherwise, if in a given period it was not possible to serve the entire demand by  $r$ -type agents, then the solution to the platform's problem would simply consist of serving  $r$ -type agents only and charge them a price of 1.

### 3.3. Initial Results and Problem Transformation

We now turn to a set of initial results shown in Proposition 2 that outline important properties of the optimal mechanism. It will also allow to derive a simplified formulation of Problem ( $\mathcal{P}$ ).

PROPOSITION 2. *The optimal solution to problem ( $\mathcal{P}$ ) satisfies, for each state variable  $(D, \Gamma)$ :*

- (i)  $p_t(D, \Gamma) = v_1$ , and  $p_l(D, \Gamma) = v_2$ .
- (ii)  $q_r(D, \Gamma) = 1$ , and  $p_r(D, \Gamma) \geq v_1$ .
- (iii) If  $\Gamma > 0$ , then  $q_l(D, \Gamma) = \min \left\{ 1, \frac{1 - \sigma D - q_t(D, \Gamma)(1 - \sigma)D}{\Gamma} \right\}$ .
- (iv) Constraint  $\mathcal{IC}_r$  is binding, and hence  $p_r(D, \Gamma) = 1 - q_t(D, \Gamma)(1 - v_1)$ .

The first part of Proposition 2 asserts that the expected utility of an  $n$ -type agent is always zero, i.e.,  $p_t = v_1$  and  $p_l = v_2$ . This is expected, as any smaller price would induce a revenue loss for the platform without altering the incentives of any agent. As a result, in the optimal mechanism, the platform extracts all of the surplus from  $n$ -type agents.

The second part of Proposition 2 states that the price  $p_r$  is at least equal to  $v_1$ . This is because any price lower than  $v_1$  would violate the incentive compatibility constraint of the  $n$ -type agents, who could then report their type as  $r$  and receive a positive expected utility. Constraint  $\mathcal{IC}_n$  is thus automatically satisfied. The result also indicates that the time-sensitive agents are always given a timely service with probability 1, which is feasible due to Assumption 1. This is also intuitive, as the price charged to the  $r$ -type agents is always greater than the price charged to the  $n$ -type agents, so the platform cannot benefit from refusing service to the  $r$ -type agents. Note, nonetheless, that the price  $p_r$  may be strictly lower than 1 in order to satisfy Constraint  $\mathcal{IC}_r$ . In other words, the platform might not be able to extract all the surplus from the time-sensitive agents.

The third part of Proposition 2 asserts that, once the platform allocates all the timely services (which amounts to  $\sigma D$  for  $r$ -type agents and  $q_t(D, \Gamma)(1 - \sigma)D$  for  $n$ -type agents), the remaining suppliers are matched with the  $n$ -type agents transferred from the previous period. Indeed, there is no reason for the platform to keep available capacity idle when people are waiting for a late service, as it would induce a revenue loss without affecting the incentives of the  $r$ -type agents. Nonetheless, some late services may be rejected due to insufficient supply.

Finally, the last part of the result indicates that, in an optimal mechanism, the incentive constraint of the  $r$ -type agents  $\mathcal{IC}_r$  is always binding. Indeed, otherwise, the platform could simply increase the price  $p_r$  to increase its profit. Given that  $q_r(D, \Gamma) = 1$  and  $p_t(D, \Gamma) = v_1$ , we obtain  $p_r(D, \Gamma) = 1 - q_t(D, \Gamma)(1 - v_1)$ . As the probability  $q_t(D, \Gamma)$  of receiving a timely service for an  $n$ -type agent increases, misreporting becomes more attractive for an  $r$ -type agent, so the platform needs to charge a lower price  $p_r$  to ensure incentive compatibility. Conversely, as  $v_1$  increases, the price charged to  $n$ -type agents for timely services increases, so misreporting becomes less desirable

for  $r$ -type agents, and  $p_r$  increases. Since  $p_r$  is the only pricing decision of the platform, we refer to it simply as “price” in the remainder of this paper.

From Proposition 2, we know the values of  $p_t(D, \Gamma)$ ,  $p_l(D, \Gamma)$ , and  $q_r(D, \Gamma)$ , as well as  $q_l(D, \Gamma)$  and  $p_r(D, \Gamma)$  as a function of  $q_t(D, \Gamma)$ . Hence, the optimal mechanism is fully characterized by the value of  $q_t(D, \Gamma)$ , which is itself determined by the value of  $\Gamma'$  (Equation (9)). We thus reformulate the platform’s problem with the single decision variable  $\Gamma'$ , defined as the number of  $n$ -type agents transferred to the next period. Before proceeding further, we derive lower and upper bounds for  $\Gamma'$ . Note, first, that  $\Gamma' \leq (1 - \sigma)D$  for any demand realization  $D$ , because no more  $n$ -type agents can be transferred to the next period than the number of  $n$ -type agents that arrived in the current period. Moreover, we obtain that  $\Gamma' \geq D - 1$  from Equations (8) and (9) by using the fact that  $q_r = 1$  (Proposition 2). Therefore,  $\Gamma'$  is at least equal to  $H - 1$  under high demand, since it is not feasible to assign a timely service to all  $n$ -type agents. In contrast, under low demand, the platform can feasibly provide timely services to all agents. This is summarized as follows:

If  $D = H$ , then  $\Gamma' \in [\underline{\Gamma}_H, \bar{\Gamma}_H]$ , where  $\bar{\Gamma}_H = (1 - \sigma)H$  and  $\underline{\Gamma}_H = H - 1$ .

If  $D = L$ , then  $\Gamma' \in [\underline{\Gamma}_L, \bar{\Gamma}_L]$ , where  $\bar{\Gamma}_L = (1 - \sigma)L$ , and  $\underline{\Gamma}_L = 0$ .

The *optimal policy function* is denoted by  $\Gamma^*(D, \Gamma)$ , where  $\Gamma^*(D, \Gamma) \in [\underline{\Gamma}_D, \bar{\Gamma}_D]$  for each  $D \in \{H, L\}$ . We also denote by  $\mathbf{\Gamma} = [\underline{\Gamma}_H, \bar{\Gamma}_H] \cup [\underline{\Gamma}_L, \bar{\Gamma}_L]$ .

In any state  $(D, \Gamma)$  and for an arbitrary choice of  $\Gamma' \in [\underline{\Gamma}_D, \bar{\Gamma}_D]$ , let  $\tilde{V}(D, \Gamma, \Gamma')$  be the value of the platform’s objective function, given that the optimal policy will be applied from the following period onward. Note that, for a given choice of  $\Gamma'$ , the number of  $n$ -type agents that are provided a timely service is equal to  $(1 - \sigma)D - \Gamma'$ . From Proposition 2, we know that  $r$ -type agents are charged a price  $p_r = 1 - \frac{(1-\sigma)D-\Gamma'}{(1-\sigma)D}(1 - v_1)$  and that any  $n$ -type agent transferred from the previous period is served with probability  $q_l = \min\left\{1, \frac{1-D+\Gamma'}{\Gamma}\right\}$ . Putting it all together, and leveraging the facts that  $q_r = 1$ ,  $p_t = v_1$  and  $p_l = v_2$ , we have:

$$\begin{aligned} \tilde{V}(D, \Gamma, \Gamma') = & \underbrace{\sigma D \left(1 - \frac{(1-\sigma)D-\Gamma'}{(1-\sigma)D}(1 - v_1)\right)}_{r\text{-type agents}} + \underbrace{((1 - \sigma)D - \Gamma')v_1}_{\text{timely services, } n\text{-type agents}} + \underbrace{\min\{\Gamma, 1 - D + \Gamma'\}v_2}_{\text{late services, } n\text{-type agents}} \\ & + \underbrace{\delta(kV(H, \Gamma') + (1 - k)V(L, \Gamma'))}_{\text{future value}}. \end{aligned} \quad (10)$$

Note that all the constraints listed in Problem  $(\mathcal{P})$  are embedded into this expression. We can then formulate the platform’s dynamic problem with the decision variable  $\Gamma' \in [\underline{\Gamma}_D, \bar{\Gamma}_D]$ , which we refer to as Problem  $(\hat{\mathcal{P}})$ . Since the existence of the solution for  $(\mathcal{P})$  is guaranteed by Proposition 1, we are also guaranteed to have a solution for Problem  $(\hat{\mathcal{P}})$ .

$$V(D, \Gamma) = \max_{\Gamma' \in [\underline{\Gamma}_D, \bar{\Gamma}_D]} \tilde{V}(D, \Gamma, \Gamma'). \quad (\hat{\mathcal{P}})$$

PROPOSITION 3. *The value  $V(D, \Gamma)$  is a non-decreasing and concave function of  $\Gamma$ , and is differentiable almost everywhere with respect to  $\Gamma$ .*

## 4. Characterizing the Optimal Policy

We first derive structural insights on the main trade-offs that govern the platform's choice. We derive a closed-form expression of the optimal policy across the full parameter space, which we partition into three regions based on the heterogeneity across agents and the time preferences of the price-sensitive agents. We conclude by presenting the steady-state dynamics of the system.

### 4.1. Preliminaries

To simplify the exposition, we denote by  $\zeta(D, \Gamma)$  the number of suppliers that are not assigned to a timely service in the optimal mechanism, i.e.,

$$\zeta(D, \Gamma) = 1 - D + \Gamma^*(D, \Gamma).$$

From Proposition 2, we know that the total number of late services provided in a period is then equal to  $\min\{\Gamma, \zeta(D, \Gamma)\}$ . If  $\zeta(D, \Gamma)$  is larger than  $\Gamma$ , the platform leaves some of its capacity idle, either because the overall effective demand (for timely and late services) is less than capacity, or because the platform may elect to deliberately restrict the number of timely services provided to  $n$ -type agents for discriminatory purposes (i.e., to charge a higher price to the  $r$ -type agents). We also denote by  $\tilde{\zeta}(D, \Gamma, \Gamma') = 1 - D + \Gamma'$  the analog of  $\zeta(D, \Gamma)$  that arises from an arbitrary choice of  $\Gamma' \in [\underline{\Gamma}_D, \bar{\Gamma}_D]$ .

We denote by  $V'(D, \Gamma)$  the partial derivative of  $V$  with respect to  $\Gamma$  (which exists almost everywhere from Proposition 3). We obtain from Equation (10), for each  $(D, \Gamma) \in \{H, L\} \times \mathbf{\Gamma}$ :

$$\frac{\partial \tilde{V}(D, \Gamma, \Gamma')}{\partial \Gamma'} = \underbrace{\frac{\sigma(1-v_1)}{1-\sigma}}_{r\text{-type agents}} - \underbrace{v_1}_{\substack{\text{timely services} \\ n\text{-type agents}}} + \underbrace{\frac{\partial \min\{\Gamma, 1-D+\Gamma'\}v_2}{\partial \Gamma'}}_{\substack{\text{late services, } n\text{-type agents}}} + \underbrace{\delta(kV'(H, \Gamma') + (1-k)V'(L, \Gamma'))}_{\text{inter-temporal Effects}}. \quad (11)$$

The effect of an infinitesimal change in  $\Gamma' \in [\underline{\Gamma}_D, \bar{\Gamma}_D]$  on the platform's expected profit comprises four components. The first term reflects the *positive* effect on the profit raised from the  $r$ -type agents, since higher  $\Gamma'$  implies a higher price  $p_r$  due to the incentive compatibility constraint. The second term reflects the *negative* effect from the timely services provided to the  $n$ -type agents, since a higher  $\Gamma'$  implies a quantity loss and thus a profit loss at rate  $v_1$  (i.e., the price of timely services). The third term reflects the *non-negative* effect from the late services. As  $\Gamma'$  increases, less capacity are used to provide timely services, so the platform can serve a (weakly) higher number of late requests. If  $\Gamma > \tilde{\zeta}(D, \Gamma, \Gamma')$ , some of the demand for late services ("late demand", henceforth) remains unserved, so increasing  $\Gamma'$  enables the platform to provide additional late services, which

increases its profit at rate  $v_2$  (i.e., the price of late services). Conversely, if  $\Gamma \leq \tilde{\zeta}(D, \Gamma, \Gamma')$ , all the late demand is served, so marginal changes in  $\Gamma'$  have no effect on the profit contribution of the late services. The last term corresponds to the *non-negative* inter-temporal effect, which reflects that higher values of  $\Gamma'$  result in higher late demand in the next period, and at least the same future value since  $V(D, \Gamma)$  is weakly increasing in  $\Gamma$ .

We now identify some general properties of the optimal policy function. First, Lemma 1 shows that the number of agents transferred to the next period weakly increases in both components of the effective demand. Lemma 2 asserts that, if the platform keeps capacity idle at the optimum, then decreasing the late demand does not change the number of agents transferred to the next period. In other words, if the platform does not utilize its full capacity, then the optimal policy does not depend on marginal deviations in the late demand.

LEMMA 1. *The optimal policy function  $\Gamma^*(D, \Gamma)$  is a weakly increasing function in  $D$  and  $\Gamma$ .*

LEMMA 2. *For a given state  $(D, \Gamma_0) \in \{H, L\} \times \Gamma$ , if the optimal policy function satisfies  $\zeta(D, \Gamma_0) > \Gamma_0$ , then  $\Gamma^*(D, \Gamma) = \Gamma^*(D, \Gamma_0)$ ,  $\forall \Gamma \leq \Gamma_0$ .*

The trade-off faced by the platform can be summarized as follows. The higher the number of  $n$ -type agents that are transferred to the next period, the fewer the timely services provided to such agents, which induces a profit loss at rate  $v_1$ . However, it also increases the price charged to  $r$ -type agents, and may also result in a higher number of late services provided (which increases the profit at rate  $v_2$ ), and/or in a higher future value in the next period onward.

## 4.2. Optimal Policy

We now proceed to the characterization of the policy function  $\Gamma^*(D, \Gamma)$ . This task, however, is intricate given the model's rich set of parameters. To ensure analytical tractability, we impose two restrictions on the demand structure in Assumption 2. First, we assume that the demand is symmetric, i.e., the excess demand under high demand is equal to the excess supply under low demand. Second, we assume that the demand fluctuations are large enough, so the platform faces the two objectives of smoothing out the dynamic imbalances between demand and supply as well as price discrimination across agents. Note that, under these assumptions, the minimum number of  $n$ -type agents transferred to the next period under high demand (i.e.,  $\underline{\Gamma}_H = H - 1$ ) is sufficiently large so, if the realized demand is also high in the next period, the platform cannot provide a late service to all of these  $H - 1$  agents. This is because  $\sigma H$  suppliers will be allocated to  $r$ -type agents and  $\sigma H + H - 1 > 1$ . In other words, when demand is high in two consecutive periods, the corresponding value of the inter-temporal effect will be 0, i.e.,  $V'(H, \Gamma) = 0$  for each  $\Gamma \in [\underline{\Gamma}_H, \bar{\Gamma}_H]$ .

ASSUMPTION 2. *The parameters governing the structure of the incoming demand satisfies:*

i)  $H = 1 + d$ , and  $L = 1 - d$ , for some  $d \in (0, 1)$ .

ii)  $H - 1 > 1 - \sigma H \Leftrightarrow d > \frac{1-\sigma}{1+\sigma}$ .

As we shall see, the optimal policy exhibits different properties in three regions. Region 1 is defined as  $v_1 \leq \sigma$ , Region 2 is defined as  $\sigma < v_1 \leq \sigma + (1 - \sigma)v_2$ , and Region 3 is defined as  $v_1 > \sigma + (1 - \sigma)v_2$ . To ease the interpretation of the optimal policy, we project our parameter space into a plane where the two dimensions are (i) *inter-type heterogeneity*, measured by the dispersion across the valuations for timely services (i.e., 1 vs.  $v_1$ ) and (ii) *n-type agents' time preferences*, measured by the dispersion between their valuations from timely service vs. late service (i.e.,  $v_1$  vs.  $v_2$ ). This is shown in Figure 2.<sup>6</sup> Region 1 is defined by strong inter-type heterogeneity (i.e., large values of  $1 - v_1$ ). In contrast, Regions 2 and 3 are defined by weak inter-type heterogeneity. In this case, the time preferences of *n*-type agents play a critical role in determining the optimal policy. Region 2 is characterized by comparatively weak time preferences (i.e., small values of  $v_1 - v_2$ ), while Region 3 is characterized by strong time preferences.

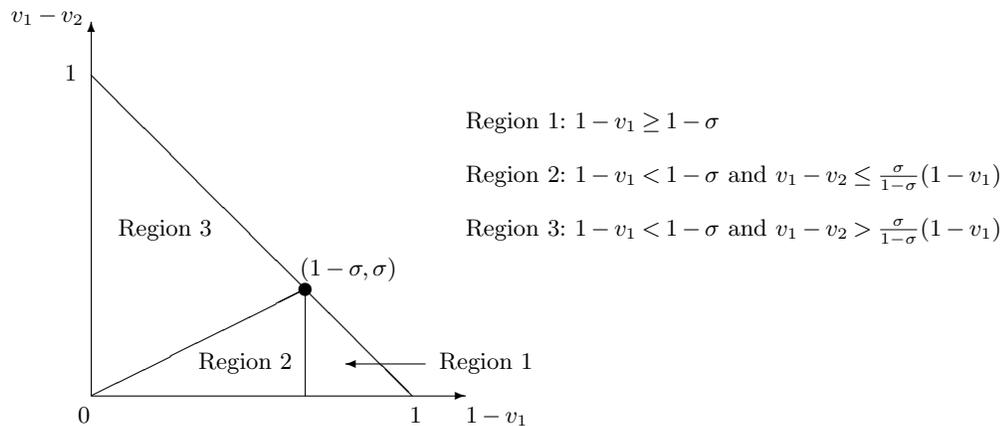


Figure 2 Definition of Region 1, Region 2 and Region 3

### Region 1: Strong inter-type heterogeneity

Region 1 is characterized by a strong differential between the valuations of timely services across agents. This creates incentives for the platform to adopt an extreme form of discrimination by providing timely services only to the *r*-type agents. All the requests from *n*-type agents are transferred to the next period, when they may be assigned a late service, or not. In this case, the *inter-type* heterogeneity is such that the time preferences of *n*-type agents do not impact the optimal policy.

Formally, we have from Equation (11), for all  $(D, \Gamma) \in \{H, L\} \times \mathbf{\Gamma}$  and  $\Gamma' \in [\underline{\Gamma}_D, \bar{\Gamma}_D]$ :

$$\frac{\partial \tilde{V}(D, \Gamma, \Gamma')}{\partial \Gamma'} \geq \frac{\sigma(1 - v_1)}{1 - \sigma} - v_1 = \frac{\sigma - v_1}{1 - \sigma} \geq 0.$$

<sup>6</sup> Our parameter specifications restrict this plane into a triangular shape since  $v_1 - v_2 < v_1 = 1 - (1 - v_1)$ .

In other words, the negative effect associated with restricting the number of timely services provided to  $n$ -type agents is offset by the associated price increase for  $r$ -type agents. Conversely, inter-type heterogeneity is such that providing timely services to  $n$ -type agents would induce a significant price decrease, and result in a net negative effect on the platform's profit. Proposition 4 shows that the optimal policy is thus to always transfer all the incoming  $n$ -type agents to the next period.

**PROPOSITION 4.** *When  $v_1 \leq \sigma$ , the optimal mechanism is characterized by:  $\Gamma^*(D, \Gamma) = \bar{\Gamma}_D$ ,  $\forall (D, \Gamma) \in \{H, L\} \times \Gamma$ .*

The implications of this result are threefold. First, the platform might strategically keep some capacity idle for discriminatory purposes. Indeed, the number of suppliers available to provide late services is equal to  $\zeta(D, \Gamma) = 1 - \sigma D$ . In instances where  $\Gamma < 1 - \sigma D$ , the optimal policy is such that *all*  $n$ -type agents are transferred to the next period even though some suppliers would have been available to provide timely services to them. Second, the price charged to  $r$ -type agents is  $p_r^*(D, \Gamma) = 1$ . This stems directly from the fact that  $q_t^*(D, \Gamma) = 0$ . Third, and as a result, all the agents receive a zero payoff, and the platform extracts all the surplus generated without leaving any information rent. Note that this surplus extraction comes at the expense of foregone potential revenues resulting from the delay, or the rejection altogether, of the requests from  $n$ -type agents.

### **Region 2: Weak inter-type heterogeneity, weak $n$ -type time preferences**

In Region 2, the positive price effect of a marginal increase in  $\Gamma'$  does not offset, by itself, any loss in timely services provided to  $n$ -type agents, so the platform may thus provide a timely service to some  $n$ -type agents. Nonetheless, transferring  $n$ -type agents to the next period remains profitable as long as it generates more late services in the current period. Specifically, we have, from Equation (11):

$$\frac{\partial \tilde{V}(D, \Gamma, \Gamma')}{\partial \Gamma'} \geq \frac{\sigma(1 - v_1)}{1 - \sigma} - v_1 + v_2 = \frac{\sigma - v_1 + v_2(1 - \sigma)}{1 - \sigma} \geq 0, \quad \text{if } \tilde{\zeta}(D, \Gamma, \Gamma') = 1 - D - \Gamma' \leq \Gamma.$$

Therefore, the platform increases the value of  $\Gamma'$  as long as  $\tilde{\zeta}(D, \Gamma, \Gamma) \leq \Gamma$ . As a result, we have either  $\zeta(D, \Gamma) = 1 - D + \bar{\Gamma}_D$  (i.e., the platform cannot transfer more agents to the next period) or  $\zeta(D, \Gamma) \geq \Gamma$  (i.e., all the late demand is served). This property is shown in Lemma 3.

**LEMMA 3.** *In Region 2, optimal policy function  $\Gamma^*(D, \Gamma)$  satisfies  $\zeta(D, \Gamma) \geq \min\{\Gamma, 1 - D + \bar{\Gamma}_D\}$ , for each  $(D, \Gamma) \in \{H, L\} \times \Gamma$ .*

This lemma shows that, in Region 2, the platform *always* prioritizes the late services over the timely services of the  $n$ -type agents. Indeed, the platform delays service to  $n$ -type agents as long as it enables to provide more late services in the current time period. In other words, the platform first serves the time-sensitive agents, and then serves the late demand. If any capacity remains available after all these services have been provided, then the platform may provide timely services

to some of the  $n$ -type agents, but may also elect to strategically keep some capacity idle (in which case  $\zeta(D, \Gamma) > \min\{\Gamma, 1 - D + \bar{\Gamma}_D\}$ ). This decision to keep some capacity idle is similar to what we observed in Region 1. However, it occurs for distinctly different reasons. Unlike in Region 1, transferring more  $n$ -type agents than necessary to serve late demand induces a loss in the current period. Nonetheless, this may remain optimal in instances where creating a backlog of demand provides strong inter-temporal benefits. This is formalized in Equation (12).

$$\frac{\partial \tilde{V}(D, \Gamma, \Gamma')}{\partial \Gamma'} = \underbrace{\frac{\sigma(1-v_1)}{1-\sigma} - v_1}_{<0} + \underbrace{\delta(kV'(H, \Gamma') + (1-k)V'(L, \Gamma'))}_{\geq 0}, \quad \text{if } \tilde{\zeta}(D, \Gamma, \Gamma') > \Gamma. \quad (12)$$

Next, Lemma 4 asserts that, if the platform does not leave idle capacity for a certain  $\Gamma_0$ , then it does not for any  $\Gamma \geq \Gamma_0$  (for the same value of the realized demand  $D$ ).

LEMMA 4. *In Region 2, if the optimal policy function satisfies  $\zeta(D, \Gamma_0) = \Gamma_0$  in some state  $(D, \Gamma_0) \in \{H, L\} \times \mathbf{\Gamma}$ , then for each  $\Gamma \geq \Gamma_0$ , we have  $\zeta(D, \Gamma) = \min\{\Gamma, 1 - D + \bar{\Gamma}_D\}$ .*

Lemmas 2, 3 and 4 together imply that the entire optimal policy function  $\Gamma^*(D, \Gamma)$  in Region 2 can be determined from  $\Gamma^*(D, 0)$ . Indeed,  $\Gamma^*(D, \Gamma)$  stays constant at  $\Gamma^*(D, 0)$  when  $\Gamma \in [0, 1 - D + \Gamma^*(D, 0)]$ , where some capacity is left idle.<sup>7</sup> Beyond that range,  $\Gamma^*(D, \Gamma)$  increases with slope 1 as a function of  $\Gamma$  as long as it is feasible. In summary, the optimal policy satisfies:

$$\Gamma^*(D, \Gamma) = \begin{cases} \Gamma^*(D, 0) & \text{if } \Gamma \leq 1 - D + \Gamma^*(D, 0), \\ \min\{\Gamma + D - 1, \bar{\Gamma}_D\} & \text{if } \Gamma \geq 1 - D + \Gamma^*(D, 0). \end{cases} \quad (13)$$

The optimal mechanism in Region 2 is elicited in Proposition 5, which indicates that Region 2 is further divided into three Sub-regions.

PROPOSITION 5. *We denote by  $v_1 = \sigma + (1 - \sigma)\delta(1 - k)v_2$  and  $\bar{v}_1 = \sigma + (1 - \sigma)\frac{\delta + \delta^2(1-k)k}{1 + \delta k}v_2$ . The optimal policy in Region 2 is characterized by the following:*

*Sub-region 2a: When  $v_1 \in (\sigma, v_1]$ ,*

$$\Gamma^*(H, \Gamma) = \begin{cases} \min\{1 - \sigma L, \bar{\Gamma}_H\} & \text{if } \Gamma \leq (1 - \sigma)L, \\ \min\{\Gamma + H - 1, \bar{\Gamma}_H\} & \text{if } \Gamma \geq (1 - \sigma)L. \end{cases} \quad \Gamma^*(L, \Gamma) = \bar{\Gamma}_L, \quad \forall \Gamma \in \mathbf{\Gamma}.$$

*Sub-region 2b: When  $v_1 \in (v_1, \bar{v}_1]$ ,*

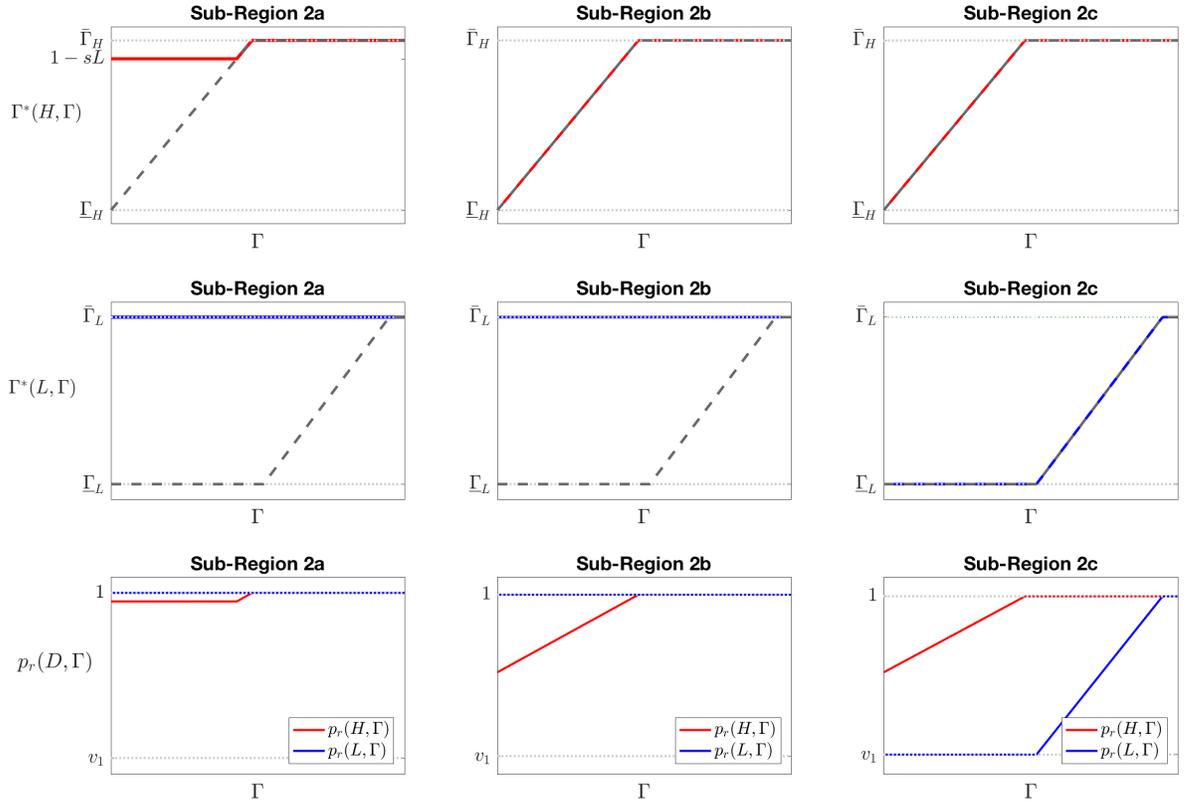
$$\Gamma^*(H, \Gamma) = \begin{cases} \Gamma + H - 1 & \text{if } \Gamma \leq 1 - \sigma H, \\ \bar{\Gamma}_H & \text{if } \Gamma \geq 1 - \sigma H. \end{cases} \quad \Gamma^*(L, \Gamma) = \min\{1 - \sigma H, \bar{\Gamma}_L\} \quad \forall \Gamma \in \mathbf{\Gamma}.$$

*Sub-region 2c: When  $v_1 \in (\bar{v}_1, \sigma + (1 - \sigma)v_2]$ ,*

$$\Gamma^*(H, \Gamma) = \begin{cases} \Gamma + H - 1 & \text{if } \Gamma \leq 1 - \sigma H, \\ \bar{\Gamma}_H & \text{if } \Gamma \geq 1 - \sigma H. \end{cases} \quad \Gamma^*(L, \Gamma) = \begin{cases} 0 & \text{if } \Gamma \leq 1 - L, \\ \min\{\Gamma + L - 1, \bar{\Gamma}_L\} & \text{if } \Gamma \geq 1 - L. \end{cases}$$

<sup>7</sup> This initial range is just a single point in case  $D = H$  and  $\Gamma^*(H, 0) = \bar{\Gamma}_H$ .

Proposition 5 together with Assumption 2 suggest that the optimal policy depends on whether the  $n$ -type or the  $r$ -type agents comprise the majority of the incoming demand (i.e.,  $\sigma < 0.5$  vs  $\sigma \geq 0.5$ ). To see this, note that the two threshold values  $1 - \sigma H$  and  $(1 - \sigma)L$ , which impact the structure of the optimal policy, satisfy  $1 - \sigma H \leq (1 - \sigma)L \iff \sigma \geq 0.5$  (Assumption 2). To simplify the exposition, we focus on the case with  $\sigma < 0.5$ . Figure 3 shows the optimal policy under high and low demand (first and second rows, respectively) and the corresponding price function  $p_r(D, \Gamma)$  (third row) over Sub-regions 2a, 2b and 2c (first, second and third columns, respectively). In the first two rows, the horizontal dotted lines correspond to the maximum and the minimum values of  $\Gamma^*(D, \Gamma)$ , i.e.,  $\bar{\Gamma}_D$  and  $\underline{\Gamma}_D$ , and the dashed line shows the minimal number of agents that need to be transferred to the next period to serve the entire late demand. From Lemma 3, we know that the optimal policy  $\Gamma^*(D, \Gamma)$  always lies on the top of this dashed line, which reflects that the platform prioritizes the late services over the timely services for  $n$ -type agents.



**Figure 3** Optimal policy function  $\Gamma^*(D, \Gamma)$  and the corresponding price  $p_r(D, \Gamma)$  in Region 2, when  $\sigma < 0.5$ .

First, in Sub-region 2a,  $v_1$  is still small, hence the platform chooses to discriminate over the heterogenous agents. Similar to Region 1, it strategically keeps some of its capacity idle for dis-

crimatory purposes, when  $\Gamma$  is small enough. To see this, note that the optimal policy function lies strictly above the dashed line, which reflects that more  $n$ -type agents are transferred to the next period than the amount needed to serve the late demand. However, unlike in Region 1, we do not observe the extreme form of discrimination. Indeed, some timely services are provided to  $n$ -type agents under high demand and low values of  $\Gamma$ . In Sub-region 2b, as the value of  $v_1$  becomes larger, discrimination across agents becomes less desirable. The platform never keeps idle capacity for discriminatory purposes under high demand. In contrast, the mechanism still keeps some capacity strategically idle for discriminatory purposes under low demand. Finally, in Sub-region 2c, the value of  $v_1$  is such that the platform never transfers more  $n$ -type agents than necessary to serve the late demand, regardless of the realized demand.

Turning to the prices, note, first, that the optimal price weakly increases with the late demand ( $\Gamma$ ). This is because higher values of  $\Gamma$  result in (weakly) higher numbers of  $n$ -type agents transferred to the next period (Lemma 1), which implies (weakly) higher prices charged to  $r$ -type agents (Proposition 2). One would expect a similar monotonic relationship with respect to the realized demand ( $D$ ). In fact, in Sub-region 2c, we do observe that the price in a high-demand period is uniformly (weakly) higher than the price in a low-demand period. However, this is *not* the case in Sub-regions 2a and 2b. In other words, there exist instances where, surprisingly, the price charged to the  $r$ -type agents is strictly higher under low demand than under high demand.

To see the intuition behind this, note that, in Sub-regions 2a and 2b, the platform transfers all the  $n$ -type agents to the next period under low demand. This is motivated by the platform's opportunity to charge a higher price to the  $r$ -type agents, as well as its expectation that it will serve this demand by providing late services in the next period. Under high demand, in contrast, transferring all the  $n$ -type agents would result in excessive late demand in the next period, which the platform may not be able to serve entirely (especially if demand is also high in the next period). To avoid this situation, the platform elects to provide some timely services to the  $n$ -type agents under high demand. This results in a lower price charged to the  $r$ -type agents than under low demand, due to the incentive compatibility constraint.

One additional takeaway from Figure 3 is that, unlike in Region 1, the platform does not always set a price  $p_r = 1$  equal to the  $r$ -type agents' willingness to pay. This implies that it is not able to extract all the surplus generated, and has to leave some information rent to the  $r$ -type agents.

### **Region 3: Weak inter-type heterogeneity, strong $n$ -type time preferences**

In Region 3, as in Region 2, the positive price effect of a marginal increase in  $\Gamma'$  does not offset the loss in timely services provided to  $n$ -type agents. However, this loss is so large that it cannot be offset even if it generates more late services in the current period. This stems from the fact

that, in Region 3, the first three terms of Equation (11) always sum up to a negative value. In a myopic setting with  $\delta = 0$ , the platform would set  $\Gamma^*(D, \Gamma)$  to its minimal value  $\underline{\Gamma}_D$ . When  $\delta > 0$ , the platform might still set  $\Gamma^*(D, \Gamma) > \underline{\Gamma}_D$  if the future benefits (from a higher late demand) offset the profit loss in the current period. Therefore, unlike in Region 2 where late services were always prioritized over timely services for  $n$ -type agents, timely services might be prioritized over late services for  $n$ -type agents in Region 3 when the inter-temporal effects are small enough.

We now characterize the structure of the optimal policy in more detail. First, Lemma 5 shows that, if the platform decides to provide so many timely services to  $n$ -type agents that some late demand remains unserved, the number of suppliers allocated to provide timely and late services remains invariant as  $\Gamma$  increases. This is because the inter-temporal effect is too small to offset the loss in the current period associated with transferring more  $n$ -type agents.

LEMMA 5. *In Region 3, if the optimal policy satisfies  $\zeta(D, \Gamma_0) < \Gamma_0$  in some state  $(D, \Gamma_0) \in \{H, L\} \times \mathbf{\Gamma}$ ; then for each  $\Gamma \geq \Gamma_0$  we must have  $\Gamma^*(D, \Gamma) = \Gamma^*(D, \Gamma_0)$ .*

Next, Lemma 6 indicates that, if there is no late demand, the platform provides as many timely services as possible (i.e.,  $\Gamma^*(D, \Gamma)$  is at its minimum value). This stems from the fact that transferring agents to the next period would not generate more late services in the current period, and, hence, the inter-temporal effect can never be large enough to offset the associated loss.

LEMMA 6. *In Region 3, the optimal policy satisfies  $\Gamma^*(D, 0) = \underline{\Gamma}_D$ , for each  $D \in \{H, L\}$ .*

Putting these results together, we have, under high demand,  $\Gamma^*(H, 0) = \underline{\Gamma}_H = H - 1$  (Lemma 6), and then the optimal policy either stays constant over the entire range of  $\Gamma$ , or increases with a slope 1 until it reaches a point after which it stays constant (Lemmas 2 and 5). Similarly, under low demand,  $\Gamma^*(L, 0) = \underline{\Gamma}_L = 0$ , and  $\Gamma^*(L, \Gamma)$  either stays constant over the entire range of  $\Gamma$ , or increases with a slope 1 when  $\Gamma > 1 - L$  until it reaches a point after which it stays constant. Proposition 6 leverages these results to elicit the optimal policy in Region 3.

PROPOSITION 6. *We denote by  $\underline{v}_1 = \sigma + (1 - \sigma) \frac{1 + \delta(1-k) + \delta^2(1-k)^2}{1 + \delta(1-k)} v_2$  and  $\bar{v}_1 = \sigma + (1 - \sigma)(1 + \delta(1 - k))v_2$ . The optimal policy in Region 3 is characterized by the following:<sup>8</sup>*

*Sub-region 3a: When  $v_1 \in (\sigma + (1 - \sigma)v_2, \underline{v}_1]$ ,*

$$\Gamma^*(H, \Gamma) = \begin{cases} \min\{\Gamma + H - 1, \bar{\Gamma}_H\} & \text{if } \Gamma \leq (1 - \sigma)L, \\ \min\{1 - \sigma L, \bar{\Gamma}_H\} & \text{if } \Gamma \geq (1 - \sigma)L. \end{cases} \quad \Gamma^*(L, \Gamma) = \begin{cases} 0 & \text{if } \Gamma \leq 1 - L, \\ \min\{\Gamma + L - 1, \bar{\Gamma}_L\} & \text{if } \Gamma \geq 1 - L. \end{cases}$$

*Sub-region 3b: When  $v_1 \in (\underline{v}_1, \bar{v}_1]$ ,*

<sup>8</sup> Note that the three Sub-regions are not guaranteed to exist, depending of the parameter values. In order to have a consistent exposition, we assume in the discussion that  $\bar{v}_1 < 1$ .

$$\Gamma^*(H, \Gamma) = \underline{\Gamma}_H, \quad \forall \Gamma. \qquad \Gamma^*(L, \Gamma) = \begin{cases} 0 & \text{if } \Gamma \leq 1 - L, \\ \min\{\Gamma + L - 1, \bar{\Gamma}_L\} & \text{if } \Gamma \geq 1 - L. \end{cases}$$

*Sub-region 3c: When  $v_1 \in (\bar{v}_1, 1)$ ,*

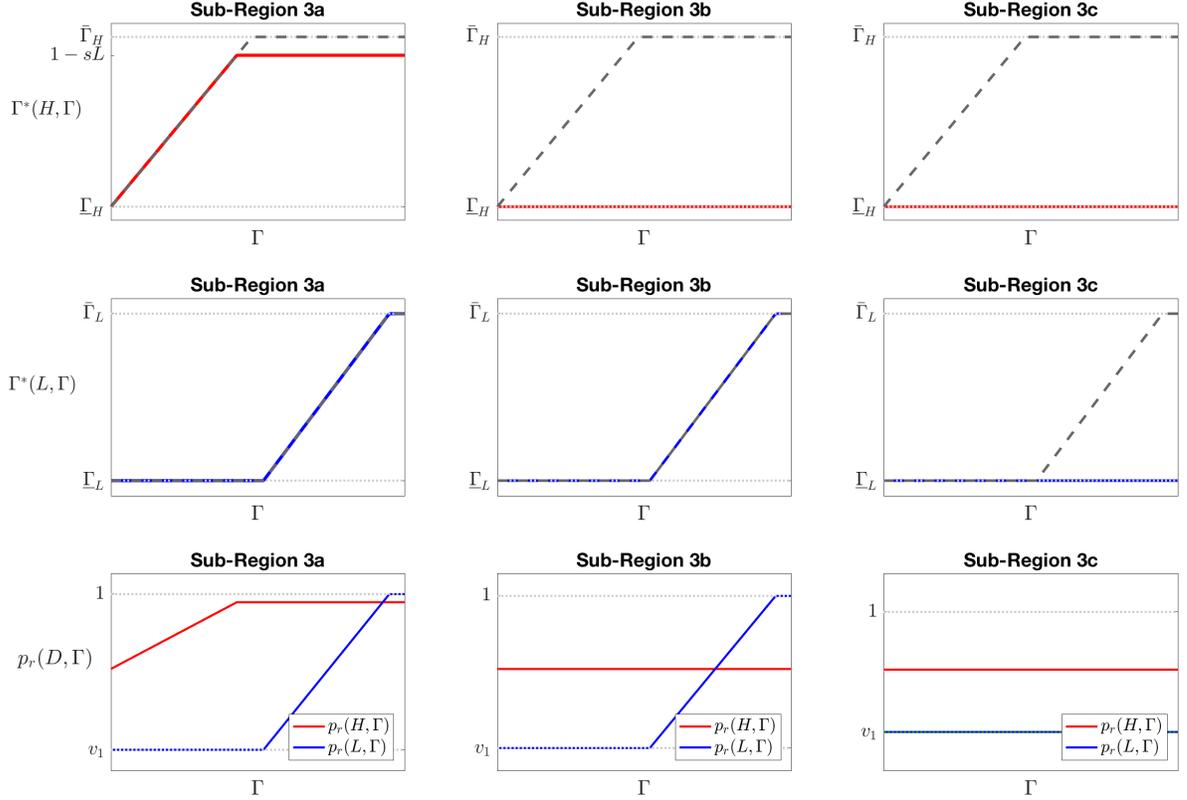
$$\Gamma^*(H, \Gamma) = \underline{\Gamma}_H, \quad \forall \Gamma. \qquad \Gamma^*(L, \Gamma) = \underline{\Gamma}_L, \quad \forall \Gamma.$$

Figure 4 illustrates the optimal policy and the price charged to  $r$ -type agents over Region 3 when  $\sigma < 0.5$ , using the same nomenclature as in Figure 3. The main difference from Region 2 is that  $\Gamma^*(D, \Gamma)$  always lies on or below the dashed line. This reflects the fact that the platform never transfers more agents to the next period than required to serve the late demand in the current period. In other words, unlike in Regions 1 and 2, the platform never keeps any capacity idle strategically for discriminatory purposes.<sup>9</sup> When the realized demand  $D$  or the late demand  $\Gamma$  is large enough, some late demand is actually left unserved. In these cases, if the platform were to serve these late services, it would have to transfer more  $n$ -type agents to the next period, but the resulting late demand in the next period would be so high that it would likely not be able to serve all of it. Therefore the loss associated with the transfer of  $n$ -type agents would not be offset by the corresponding inter-temporal benefits.

Specifically, in Sub-region 3a, where the value of  $v_1$  is the lowest, the platform continues to serve the late demand until the number of  $n$ -type agents transferred to the next period reaches  $1 - \sigma L$ . This value corresponds to the threshold above which the platform will surely not be able to serve the late demand in the subsequent period, regardless of the demand realization. In Sub-region 3b, the platform utilizes its capacity to provide timely services exclusively under high demand, but provides late services under low demand even if it involves transferring more  $n$ -type agents to the next period. Finally, in Sub-region 3c, heterogeneity across agents is so low that the platform fully prioritizes timely services regardless of the demand realization. Late services are only provided only under low demand, when the platform faces an excess supply.

On the pricing side, the three main takeaways from Figure 4 are consistent with the observations in Region 2. First, the price  $p_r$  charged to  $r$ -type agents (weakly) increases with the late demand  $\Gamma$ . Second, in most cases, the price charged to  $r$ -type agents is strictly lower than 1, i.e., the platform does *not* extract the full surplus generated and leaves some information rent to the  $r$ -type agents. This stems from the fact that, given the weak inter-type heterogeneity in Region 3, the platform focuses on smoothing out the imbalances between demand and supply rather than on discrimination across agents. Third, and most importantly, the price charged to the  $r$ -type agents

<sup>9</sup> When  $D = L$  and  $\Gamma \leq 1 - L$ , some capacity is idle, but this stems from excess capacity rather than discrimination.



**Figure 4** Optimal policy function  $\Gamma^*(D, \Gamma)$  and the corresponding price  $p_r(D, \Gamma)$  in Region 3, when  $\sigma < 0.5$ .

does not necessarily increase with the realized demand. Specifically, in Sub-regions 3a and 3b, the price  $p_r$  is higher under low demand than under high demand for the highest values of  $\Gamma$ . The intuition underlying this result is consistent with the insights generated in Region 2. Indeed, when the late demand is large enough, the platform provides late services under low demand, and transfers  $n$ -type agents to the next period. However, under high demand, the same strategy would result in a higher late demand in the next period, which the platform might not be able to serve (especially if the demand is also high in the next period). Therefore, the platform rejects some of the late demand and provides more timely services to  $n$ -type agents instead, which results in a lower price charged to the  $r$ -type agents under high demand than under low demand due to the incentive compatibility constraints.

### 4.3. Steady State and Transition Dynamics

In Section 4.2, we have fully characterized the optimal mechanism over Region 1, Region 2 and Region 3. We now derive the steady-state dynamics of the optimal mechanism, as well as aggregate pricing and allocation metrics. This will be used in Section 5 to quantify the surplus generated

within the platform and its distribution among the participants, and to compare this mechanism against the first-best allocation rule and a baseline surge pricing policy.<sup>10</sup>

Figure 5 shows the steady-state distribution and the transition dynamics of the optimal mechanism over the regions characterized in Propositions 4, 5 and 6. This representation is based on a directed graph, where each node denotes a value of  $\Gamma$  supported by the steady state distribution, and each edge denotes a demand realization  $D = H$  or  $D = L$  (which occurs with probability  $k$  or  $1 - k$ , respectively). In other words, the state  $(D, \Gamma)$  is captured by each directed edge together with its initial node, and the optimal policy  $\Gamma'$  is captured by the terminal node of the edge. The steady-state probability of each value of  $\Gamma$  is depicted within the corresponding node.<sup>11</sup> We find that  $\Gamma$  can take at most five different values in the steady state:  $\bar{\Gamma}_L = 0$ ,  $\bar{\Gamma}_L = (1 - \sigma)L$ ,  $\bar{\Gamma}_H = H - 1$ ,  $1 - \sigma L$ , and  $\bar{\Gamma}_H = (1 - \sigma)H$ . Each of these five values is supported by the steady-state distribution in Sub-region 2c, but only a subset of them is attained in the other regions.

In Region 1,  $\Gamma$  can take only two steady-state values,  $\bar{\Gamma}_H$ , and  $\bar{\Gamma}_L$ , with respective probabilities  $k$  and  $1 - k$ . This stems from the fact that the platform always transfers the maximal number of  $n$ -type agents to the next period. The optimal mechanism is thus time-independent, as it only depends on the realized demand in the current period. This is reflected in Figure 5a by the fact that all the edges corresponding to  $D = H$  and to  $D = L$  have the same terminal node.

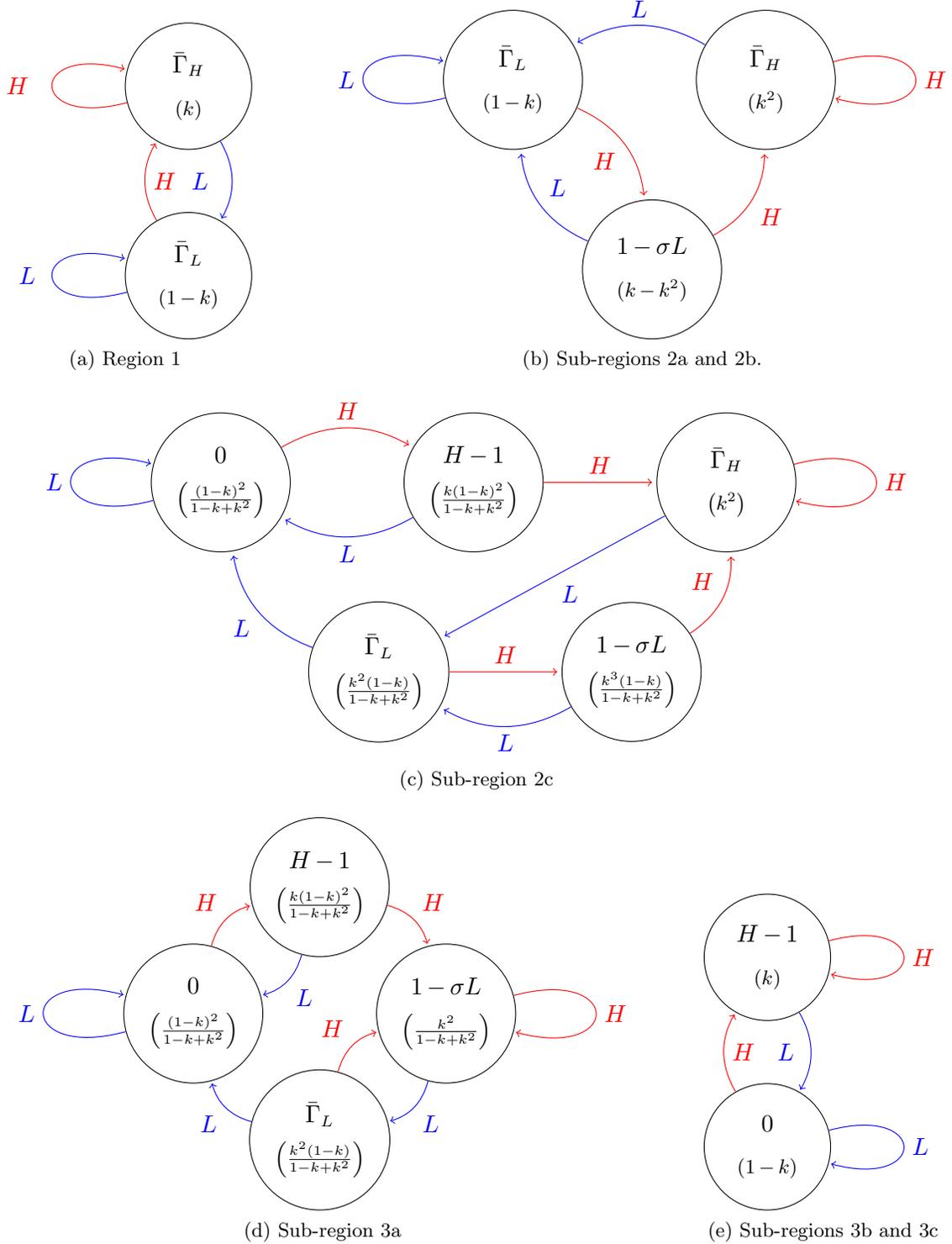
We now turn to Sub-regions 2a and 2b (Figure 5b). The steady-state distribution of  $\Gamma$  supports exactly three values:  $\bar{\Gamma}_L$ ,  $1 - \sigma L$  and  $\bar{\Gamma}_H$ .<sup>12</sup> Under low demand (which occurs with probability  $1 - k$ ), we have  $\Gamma^*(L, \Gamma) = \bar{\Gamma}_L$  for all  $\Gamma$ , i.e., the platform transfers all the  $n$ -type agents to the next period. Under high demand, however, we have  $\Gamma^*(H, \bar{\Gamma}_L) = 1 - \sigma L$ , and  $\Gamma^*(H, 1 - \sigma L) = \bar{\Gamma}_H$ . In other words, the value of  $\Gamma = 1 - \sigma L$  (resp.  $\Gamma = \bar{\Gamma}_H$ ) takes place after a succession of a low-demand period and a high-demand period (resp. two high-demand periods), and thus occurs with probability  $(1 - k)k$  (resp.  $k^2$ ). Unlike in Region 1, the steady state dynamics exhibit history dependence in high demand periods, as the optimal policy differs depending on whether demand in the previous period was high (in which case the platform transfers all the  $n$ -type agents to the next period) or low (in which case the platform only transfers  $1 - \sigma L$  agents).

In Sub-region 2c (Figure 5c), the optimal policy exhibits history dependence both under high demand and low demand. In other words, the optimal pricing and allocation policies depend on

<sup>10</sup> As in Section 4.2, we focus on the case where the price-sensitive agents comprise the majority of demand, i.e.,  $\sigma < 0.5$ . The case where  $\sigma \geq 0.5$  is similar, so this restriction does not induce any loss of insights.

<sup>11</sup> Let  $\{\Gamma_1^{ss}, \dots, \Gamma_n^{ss}\}$  be the set of steady-state values of  $\Gamma$ ,  $\rho_1, \dots, \rho_n$  be the steady-state probability of the system being in state  $\Gamma_i^{ss}$ , and  $\tau_{ij}$  be the transition probability from  $\Gamma_i^{ss}$  to  $\Gamma_j^{ss}$ . This steady-state distribution is obtained by solving the system of equations  $\rho_i = \sum_{j=1}^n \rho_j \tau_{ji}$ ,  $\forall i \in \{1, \dots, n\}$ .

<sup>12</sup> This can be verified by noting that, for any value of  $\Gamma$ , the optimal policy will transition into one of these values.



**Figure 5** Steady state transition dynamics, with  $\sigma < 0.5$ .

the demand realizations in the current period as well as the previous periods. As the platform prioritizes late services in Region 2, it does not provide any timely service to the  $n$ -type agents if the late demand is large enough. This occurs when  $\Gamma \in \{H - 1, 1 - \sigma L, \bar{\Gamma}_H\}$  under high demand,

and when  $\Gamma \in \{1 - \sigma L, \bar{\Gamma}_H\}$  under low demand. Vice versa, when  $\Gamma$  is smaller than these values, the platform provides timely services to  $n$ -type agents with some positive probability. We observe similar dynamics in Sub-region 3a (Figure 5d), except that the platform never transfers more than  $1 - \sigma L$   $n$ -type agents to the next period. In Sub-regions 3b and 3c, in contrast, the platform fully prioritizes timely services and transfers the minimal number of agents to the next period. In this case, the optimal policy is history-independent as in Region 1.

We conclude by reporting the corresponding probabilities of the allocation of timely service for  $n$ -type agents ( $q_t$ ), and the price charged from  $r$ -type agents ( $p_r$ ) over the full steady-state distribution in Table 1. First, note that as the value of  $v_1$  increases, the optimal mechanism allocates more timely services to  $n$ -type agents, and, as a result, the price charged to the  $r$ -type agents decreases. At one extreme, in Region 1, we have  $q_t = 0$  and  $p_r = 1$ . At the other extreme, in Sub-region 3c, we have  $q_t = 1$  and  $p_r = v_1$  as long as it is feasible (i.e., under low demand). Under high demand, the platform provides as many timely services as possible to the  $n$ -type agents (the corresponding value of  $q_t$  is obtained by solving  $\sigma H + (1 - \sigma)H \times q_t = 1$ ). In-between, the pricing and allocation mechanism depends on the realized demand and the late demand.

Finally, note that our critical takeaway that the price charged to the  $r$ -type agents does not monotonically increase with the value of the realized demand, elicited in Section 4.2, is actually observed in the steady state. Specifically,  $p_r$  is higher under low demand than under high demand when  $\Gamma = \bar{\Gamma}_L$  in Sub-regions 2a and 2b, and when  $\Gamma = 1 - \sigma L$  in Sub-region 3a.

**Table 1** Steady state allocations and prices, shown as  $(q_t(D, \Gamma), p_r(D, \Gamma))$ , when  $\sigma < 0.5$ .

State	Reg. 1	Reg. 2a&2b	Reg. 2c	Reg. 3a	Reg. 3b&3c
$(H, 0)$			$(q_t^\alpha, p_r^\alpha)$	$(q_t^\alpha, p_r^\alpha)$	$(q_t^\alpha, p_r^\alpha)$
$(L, 0)$			$(1, v_1)$	$(1, v_1)$	$(1, v_1)$
$(H, \bar{\Gamma}_L)$	$(0, 1)$	$(q_t^\beta, p_r^\beta)$	$(q_t^\beta, p_r^\beta)$	$(q_t^\beta, p_r^\beta)$	
$(L, \bar{\Gamma}_L)$	$(0, 1)$	$(0, 1)$	$(1, v_1)$	$(1, v_1)$	
$(H, H - 1)$			$(0, 1)$	$(q_t^\beta, p_r^\beta)$	$(q_t^\alpha, p_r^\alpha)$
$(L, H - 1)$			$(1, v_1)$	$(1, v_1)$	$(1, v_1)$
$(H, 1 - \sigma L)$		$(0, 1)$	$(0, 1)$	$(q_t^\beta, p_r^\beta)$	
$(L, 1 - \sigma L)$		$(0, 1)$	$(0, 1)$	$(0, 1)$	
$(H, \bar{\Gamma}_H)$	$(0, 1)$	$(0, 1)$	$(0, 1)$		
$(L, \bar{\Gamma}_H)$	$(0, 1)$	$(0, 1)$	$(0, 1)$		

$$q_t^\alpha = \frac{1 - \sigma H}{(1 - \sigma)H}$$

$$q_t^\beta = \frac{(1 - 2\sigma)d}{(1 - \sigma)H}$$

$$p_r^\alpha = 1 - q_t^\alpha(1 - v_1)$$

$$p_r^\beta = 1 - q_t^\beta(1 - v_1)$$

$$0 < q_t^\beta < q_t^\alpha < 1$$

$$v_1 < p_r^\alpha < p_r^\beta < 1$$

## 5. Welfare Implications

This section compares the proposed mechanism against two benchmarks: (i) the first-best solution that takes the perspective of a social planner under perfect information, and (ii) a surge pricing

mechanism that optimizes price levels dynamically but does not elicit agents' preferences, which aims to replicate prevalent practices from on-demand platforms. We compare the steady-state pricing and allocation strategies as well as the surplus generated under our proposed mechanism and these two benchmarks. First, we elicit the relative loss from information asymmetries, as compared to the first-best outcome, and identify a region where the optimal mechanism achieves the first-best outcome. Then, we assess the improvements in platform profits resulting from the optimal mechanism, as compared to the baseline surge pricing, and show that, under some circumstances, the optimal mechanism provides a Pareto improvement, i.e., it yields both a larger platform profit *and* a larger consumer surplus.

### 5.1. First-best Allocation

The first-best allocation can be interpreted in two different ways. First, it captures the optimal decisions of a social planner, whose objective is to maximize the social surplus (regardless of how it is shared among the participants). The corresponding first-best outcome involves only an allocation rule (i.e., to which agents the capacity should be allocated) without specifying any payment rule. Second, it represents equivalently the profit-maximizing decisions of the platform in a hypothetical setting where there is no information asymmetry and each agent's preference is observed by the platform. In this case, the platform can perfectly discriminate by charging a price equal to the agents' valuations (i.e., 1,  $v_1$  and  $v_2$ ).

As in the previous sections, we consider the state variable  $(D, \Gamma)$ . The difference is that no pricing decision is involved, and we characterize the first-best allocation rule by the choice variables  $q_r^f, q_t^f, q_l^f$ , which are the analogs of the variables  $q_r, q_t, q_l$  defined in Section 3. We refer to the first-best problem as  $\mathcal{P}_f$  and to the corresponding value function by  $V_f$ , defined as follows:

$$V_f(D, \Gamma) = \max_{q_r^f, q_t^f, q_l^f \in [0,1]} q_r^f \sigma D \times 1 + q_t^f (1 - \sigma) D v_1 + q_l^f \Gamma v_2 + \delta [k V_f(H, \Gamma') + (1 - k) V_f(L, \Gamma')] \quad (14)$$

$$\text{s.t. } 1 \geq q_r^f s D + q_t^f (1 - \sigma) D + q_l^f \Gamma, \quad (15)$$

$$\Gamma' = (1 - q_t^f)(1 - \sigma) D. \quad (16)$$

The major difference from Problem  $\mathcal{P}$  is that, since the first-best setting does not consider information asymmetries, there is no incentive compatibility and individual rationality constraints. Otherwise, Constraints (15) and (16) define the same resource constraint and the same transition function, respectively.

From the formulation of  $\mathcal{P}_f$ , note that the first-best mechanism involves maximizing the utilization of the capacity, weighted by the valuations 1,  $v_1$  and  $v_2$ . The social planner primarily aims to smooth out the imbalances between demand and supply. In the hypothetical case where  $\delta = 0$ , the first-best allocation rule simply prioritizes the  $r$ -type agents, then the  $n$ -type agents waiting for

a timely service, and last, the  $n$ -type agents waiting for a late service. However, when  $\delta > 0$ , the social planner may transfer some of the  $n$ -type agents to the next period to serve the late demand. The problem involves a trade-off between losing the surplus generated from  $n$ -type agents waiting for a late service, who leave the platform if their requests remain unserved, versus generating a smaller surplus by delaying service to the  $n$ -type agents waiting for a timely service.

We characterize properties of the optimal solution in Lemma 7, which is analogous to Proposition 2. Specifically, it asserts that all the  $r$ -type agents are always served, and that the social planner serves as much late demand as the remaining capacity allows, once timely services are provided to  $r$ -type and  $n$ -type agents. The proof of these properties follows from the same arguments as in the proof of Proposition 2, and is thus omitted for conciseness.

LEMMA 7. *The optimal solution to Problem  $\mathcal{P}_f$  satisfies:*

- (i)  $q_r^f(D, \Gamma) = 1$  for each  $(D, \Gamma) \in \{H, L\} \times \Gamma$ .
- (ii) If  $\Gamma > 0$ , then  $q_l^f(D, \Gamma) = \min \left\{ 1, \frac{1 - \sigma D - q_r^f(D, \Gamma)(1 - \sigma)D}{\Gamma} \right\}$ .

Following some simple algebra, this result allows us to transform the problem into a simplified form with the single decision variable  $\Gamma'$ , as was done in Section 3. The reformulated problem, referred to as  $\widehat{\mathcal{P}}_f$ , is defined as follows:

$$V_f(D, \Gamma) = \max_{\Gamma' \in [\underline{\Gamma}_D, \bar{\Gamma}_D]} \sigma D + ((1 - \sigma)D - \Gamma')v_1 + \min\{\Gamma, 1 - D + \Gamma'\}v_2 + \delta(kV_f(H, \Gamma') + (1 - k)V_f(L, \Gamma')). \quad (17)$$

The only difference with Problem  $\widehat{\mathcal{P}}$  lies in the first term. In Problem  $\widehat{\mathcal{P}}$ , this term included the price derived from the incentive compatibility constraint for  $r$ -type agents. In Problem  $\widehat{\mathcal{P}}_f$ , it includes their valuation for the service (equal to 1). We show in Proposition 7 that Problem  $\widehat{\mathcal{P}}_f$  is equivalent to Problem  $\widehat{\mathcal{P}}$  by transforming the parameter  $v_1$  into  $\tilde{v}_1 = \sigma + (1 - \sigma)v_1$ . This is obtained by equating the first two terms of Equation (10) with those of Equation (17), i.e.,  $\sigma D \left(1 - \frac{(1 - \sigma)D - \Gamma'}{(1 - \sigma)D} (1 - \tilde{v}_1)\right) + ((1 - \sigma)D - \Gamma')\tilde{v}_1 = \sigma D + ((1 - \sigma)D - \Gamma')v_1$ . Note that, since  $v_1 \in (v_2, 1)$ , we have  $\tilde{v}_1 \in (\sigma + (1 - \sigma)v_2, 1)$ , so the optimal policy and steady-state properties of Problem  $\widehat{\mathcal{P}}_f$  are identical to the ones of Problem  $\widehat{\mathcal{P}}$  in Region 3 (see Figures 5d and 5e).

PROPOSITION 7. *Problem  $\widehat{\mathcal{P}}_f$  with parameter  $v_1$  is equivalent to Problem  $\widehat{\mathcal{P}}$  with parameter  $\tilde{v}_1 = \sigma + (1 - \sigma)v_1$ . The optimal policy is identical to the optimal policy of Problem  $\widehat{\mathcal{P}}$  in Sub-Region 3a when  $v_1 \in \left(v_2, \frac{1 + \delta(1 - k) + \delta^2(1 - k)^2}{1 + \delta(1 - k)}v_2\right]$ , in Sub-Region 3b when  $v_1 \in \left(\frac{1 + \delta(1 - k) + \delta^2(1 - k)^2}{1 + \delta(1 - k)}v_2, (1 + \delta(1 - k))v_2\right]$ , and in Sub-Region 3c when  $v_1 \in ((1 + \delta(1 - k))v_2, 1)$ .*

This result suggests that the optimal policy adopted by the social planner is similar to the one obtained from the optimal mechanism under weak inter-type heterogeneity. This is a natural

consequence of the fact that only in Region 3 the optimal solution to  $\widehat{\mathcal{P}}$  uses strategic timing as a means of smoothing out the stochastic imbalances between demand and supply rather than a means of discriminating over agents with heterogeneous preferences. Vice versa, the stronger the inter-type heterogeneity, the stronger the effects of information asymmetries, and the more the optimal mechanism deviates from the first-best allocation rule.

## 5.2. Surge Pricing

We now turn to the analysis of a dynamic pricing mechanism. This mechanism allows the prices to vary dynamically over time, but does not elicit the agents' preferences and therefore charges a single price across all agents in each period. This aims to replicate, at a high level, the surge pricing policies used by a number of on-demand platforms (e.g., ride-sharing platforms). The system evolves as follows: (i) the platform posts a price in each period, (ii) each agent decides to accept the service, or not, (iii) the platform allocates capacity uniformly across all the agents who accept the service regardless of their types, and (iv) agents who decline the service may leave the platform or wait for a late service, depending on their type and history.

We formulate the optimal surge pricing problem, referred to as  $\mathcal{P}_s$ , as follows. The decision variable involves the price posted by the platform, denoted by  $p_s$ , as a function of the state variable  $(D, \Gamma)$ . By denoting the value function by  $V_s$ , we obtain the following program:

$$V_s(D, \Gamma) = \max_{p_s} p_s \times x(D, \Gamma, p_s) + \delta (kV_s(H, \Gamma') + (1 - k)V_s(L, \Gamma')) \quad (18)$$

$$\text{s.t. } x(D, \Gamma, p_s) = \begin{cases} \sigma D & p_s > v_1, \\ \min\{D, 1\} & p_s \in (v_2, v_1], \\ \min\{D + \Gamma, 1\} & p_s \leq v_2. \end{cases} \quad (19)$$

$$\Gamma' = \begin{cases} (1 - \sigma)D & p_s > v_1, \\ (1 - \sigma)D \left[ 1 - \frac{x(D, \Gamma, p_s)}{D} \right] & p_s \in (v_2, v_1], \\ (1 - \sigma)D \left[ 1 - \frac{x(D, \Gamma, p_s)}{D + \Gamma} \right] & p_s \leq v_2. \end{cases} \quad (20)$$

Equation (18) formulates the objective of maximizing the platform's profit in the current period and its future value. The former is equal to the posted price times the corresponding number of services provided to the agents, denoted by  $x(D, \Gamma, p_s)$ . Constraints (19) and (20) specify the relationship between the posted price  $p_s$ , the quantity  $x(D, \Gamma, p_s)$  and the number of requests transferred to the next period, respectively. When  $p_s > v_1$ , only the  $r$ -type agents accept the service, and the platform provides  $\sigma D$  services. Then, all  $n$ -type agents are transferred to the next period, so  $\Gamma' = (1 - \sigma)D$ . When  $v_2 < p_s \leq v_1$ , all the agents who arrived in the current period are willing to accept the service.<sup>13</sup> However, since the number of available suppliers is equal to 1, the number

<sup>13</sup> The  $n$ -type agents are forward-looking so that, even through the price level yields a positive payoff, they might still find it more profitable to reject it and wait for a late service if they anticipate a lower price in the next period. This, however, never occurs as the platform never charges a price strictly lower than  $v_2$  in the optimal solution.

of services is then equal to  $\min\{D, 1\}$ . In this case, the value of  $\Gamma'$  is equal to  $(1 - \sigma)D$  times the probability of not getting a timely service, i.e.,  $(1 - \sigma)D \left[1 - \frac{x(D, \Gamma, p_s)}{D}\right]$ . Last, if  $p_s \leq v_2$ , all the agents are willing to accept the service, so the number of services is then equal to  $\min\{D + \Gamma, 1\}$ , and  $\Gamma' = (1 - \sigma)D \left[1 - \frac{x(D, \Gamma, p_s)}{D + \Gamma}\right]$ . We identify properties of the optimal solution of  $\mathcal{P}_s$  in Proposition 8.

**PROPOSITION 8.** *A solution to Problem  $\mathcal{P}_s$  always exists, and we denote it by  $p_s^*(D, \Gamma)$ . It satisfies  $p_s^*(D, \Gamma) \in \{1, v_1, v_2\}$ . Moreover,  $p_s^*(D, \Gamma) \in \{1, v_1\}$  for each  $(D, \Gamma)$  if and only if  $v_2 \leq \max\{\sigma L, v_1 L\}$ . In this case, the optimal price function satisfies:*

- (i) *When  $v_1 \leq \sigma$ , we have  $p_s^*(D, \Gamma) = 1$ , for each  $(D, \Gamma)$ .*
- (ii) *When  $\sigma < v_1 \leq v_1$ , we have  $p_s^*(H, \Gamma) = 1$ , and  $p_s(L, \Gamma) = v_1$ , for each  $\Gamma$ .*
- (iii) *When  $v_1 > \sigma H$ , we have  $p_s^*(D, \Gamma) = v_1$ , for each  $(D, \Gamma)$ .*

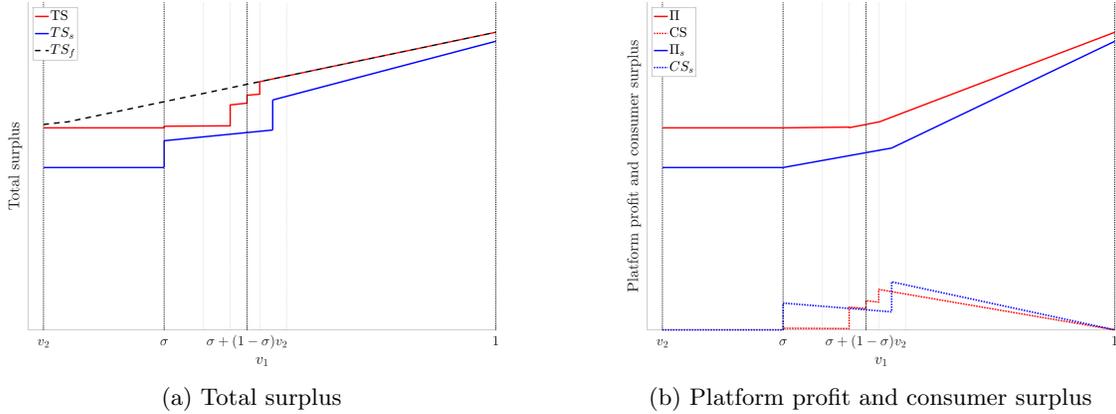
The first result of the proposition is that the optimal price levels fall into  $\{1, v_1, v_2\}$ . This is intuitive because the quantity of services accepted by the agents remains constant over  $(0, v_2]$ ,  $(v_2, v_1]$  and  $(v_1, 1]$ . We then show that, if  $v_2 \leq \max\{\sigma L, v_1 L\}$ , the value of a late service for  $n$ -type agents is sufficiently low for the platform to not to provide late services, i.e., for the platform to never charge a price of  $v_2$ . In this case, the price reduction required to provide late services offsets the corresponding increase in the quantity of services provided, so the platform elects to provide timely services. The corresponding pricing policy is then myopically optimal, i.e., Problem  $\mathcal{P}_s$  is equivalent to maximizing profitability based on the value of the realized demand in each period independently. Specifically, when  $v_2 \leq \max\{\sigma L, v_1 L\}$ , we have:

- (i) When  $v_1 \leq \sigma$ , the platform always sets a price of 1, and thus only serves the  $r$ -type agents. Inter-type heterogeneity is such that, in order to serve the  $n$ -type agents, the platform would need to reduce the price significantly and would thus be worse off. The platform's expected profit is then equal to  $\sigma(kH + (1 - k)L)$ . The expected consumer surplus is equal to 0, since the  $r$ -type agents are charged their willingness to pay and the  $n$ -type agents are not served.
- (ii) When  $\sigma < v_1 \leq \sigma H$ , the platform serves all the incoming demand in low-demand periods by setting a price of  $v_1$ , but only the  $r$ -type agents in high-demand periods by charging a price of 1. The platform's expected profit is then equal to  $k\sigma H + (1 - k)v_1 L$ . The expected consumer surplus is equal to  $k\sigma L(1 - v_1)$ , since the  $r$ -type agent are charged  $v_1$  in low-demand periods.
- (iii) When  $v_1 > \sigma H$ , the platform always sets the price of  $v_1$ . In this case, the platform cannot serve all the agents who are willing to accept the service if the realized demand is high. The platform's expected profit is equal to  $v_1(k + (1 - k)L)$  and the expected consumer surplus is equal to  $\sigma(k + (1 - k)L)(1 - v_1)$ .

### 5.3. Performance Assessment

We now compare the steady-state performance of our proposed mechanism to the one of the two benchmarks developed in this section. For analytical tractability and expositional ease, we restrict the analysis to the case where  $v_2 \leq \max\{\sigma L, v_1 L\}$ , and apply the result provided in Proposition 8. This comparison is based on three main metrics: (i) the platform’s expected profit, denoted by  $\Pi$ , (ii) the expected consumer surplus, denoted by  $CS$ , and (iii) the expected total surplus generated within the platform, denoted by  $TS$ . These three metrics are derived from the system’s steady-state probability distribution under each of the three policies. Note that, under the proposed mechanism and the surge pricing policy, only the time-sensitive agents may receive a positive surplus (Propositions 2 and 8).

Figure 6 illustrates the total surplus generated by each of the three policies (Figure 6a) and its distribution between the platform and the consumers under the proposed mechanism and the surge pricing policy (Figure 6b), as a function of the parameter  $v_1$ . By design, the optimal mechanism achieves a (weakly) lower total surplus than the first-best benchmark and a higher platform profit than the surge pricing policy. We now compare the outcomes of these policies in more detail.



**Figure 6** Surplus from the optimal mechanism, surge pricing (“s”) and the first-best mechanism (“f”)

In terms of social surplus, the optimal mechanism induces a loss as compared to the first-best outcome under strong inter-type heterogeneity, but achieves the efficient level under weak inter-type heterogeneity. For the lower values of  $v_1$  (Regions 1, 2 and 3a), the optimal mechanism focuses on discrimination across the heterogenous agents, instead of (or in addition to) smoothing the demand-supply imbalances. In some of these cases, the platform extracts all the surplus without leaving any information rent to the agents. As  $v_1$  gets closer to  $v_2$ , the relative loss of the optimal mechanism becomes smaller because the cost of strategic delays gets lower. For the highest values of  $v_1$  (Regions 3b and 3c), the optimal mechanism maximizes the total surplus because the focus

shifts from discrimination to demand-supply smoothing. In this region, the platform leaves some surplus to the agents in the form of information rent.

The main takeaways from the comparison of the optimal mechanism with surge pricing fall into three categories. First, eliciting agents' preferences and leveraging this information in the pricing and allocation policies results in a strictly larger profit for the platform than under surge pricing. In fact, the relative improvement in platform profitability can be significant, and increases with inter-type heterogeneity. For instance, when  $v_1 \leq \sigma$ , the surge pricing policy ignores all the price-sensitive agents, which obviously results in lost profitability. Second, we find that in a large majority of the cases, the optimal mechanism results in a larger total expected surplus than surge pricing.<sup>14</sup> Last, there exist settings where the optimal mechanism provides a Pareto improvement, i.e., results in larger platform profits *as well as* larger consumer surplus than surge pricing. In Figure 6b, this occurs in Region 3a, where the optimal mechanism involves weak time discrimination, while the optimal surge price is equal to 1 under high demand, which ignores the price-sensitive agents.

## 6. Conclusion

This paper designs an original pricing and allocation mechanism in the context of an on-demand platform when agents exhibit heterogeneous price-sensitivity and time-sensitivity. This paper aims to leverage this heterogeneity by enabling agents to reveal their preferences upon placing a service request, and leveraging this revealed information in setting prices and service levels.

*Strategic timing* is used as a means to: (i) smooth out the dynamic stochastic imbalances between demand and supply; and (ii) discriminate over heterogeneous agents. Under strong heterogeneity across agents, the mechanism implements an extreme form of discrimination by delaying, or rejecting, any request from price-sensitive agents to charge a higher price to time-sensitive agents. This induces a surplus loss, but the platform extracts all the surplus generated. Under weak heterogeneity across agents, the time preferences of price-sensitive agents become crucial. Under weak time preferences, the platform prioritizes the provision of late services to agents waiting in queue over new requests from price-sensitive agents. Otherwise, the platform uses strategic timing primarily to smooth out the imbalances between demand and supply rather than discrimination. In this case, the mechanism maximizes the total surplus, but the platform leaves some information rent to the time-sensitive agents.

Surprisingly, the price charged to time-sensitive agents is *not* an increasing function of the realized demand. This is because the benefits of discrimination may be lower under high demand

<sup>14</sup> There exist some parameter values where surge pricing results in a higher total surplus than the optimal mechanism in Regions 2a and 2b. This occurs when the optimal surge price is equal to  $v_1$ .

than under low demand. In the optimal mechanism, the platform may delay requests from price-sensitive agents to: (i) charge a higher price to the time-sensitive agents, and (ii) serve more of the late demand. However, this strategy would lead to a long queue when the demand is high, which the platform might not be able to meet in the future due to capacity constraints. It may then choose to serve more requests in a timely manner under high demand, which is compensated by a price reduction for the time-sensitive agents due to the incentive compatibility constraints.

Pricing remains one of the most important considerations for on-demand platforms to match demand and supply in a dynamic environment. Existing companies like Uber and Lyft use strategies like surge pricing to reach this objective. In this paper, we incorporate *timing* as another dimension that on-demand platforms can leverage to increase profitability by not only matching demand and supply, but also providing differentiated levels of service across customers that have different time preferences. Our results suggest that a mechanism that elicits such preferences can result in significant improvements in the platform's profits and even, in some instances, a higher customer surplus. This mechanism follows recent industry developments such as Uber Pool and Lyft Line, which provide differentiated services that implicitly account for heterogeneity in time preferences. In contrast, this paper formalizes this trade-off and proposes a mechanism that explicitly provides such differentiation, without resorting to the development of new products or services.

These positive results also motivate further research in this direction. First, the type of discriminatory mechanism elicited in this paper raises a number of competitive and legal questions, which we have not explicitly accounted for. Second, the focus of this paper has been on the demand-side dynamics of on-demand platforms. As a result, we abstracted away from the supply-side considerations. In practice, however, suppliers may also decide strategically when, and how to participate on the platform. Moreover, this paper has not considered quality differentiation across suppliers. For instance, in the ride-sharing context, spatial dynamics are an important contributor to service quality; in knowledge-based platforms, different suppliers may have different levels of expertise. The integration of demand-side discrimination, supply-side participation and quality differentiation represents an important research opportunity.

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## Appendix A: Proof on Pricing and Allocation Mechanism

In this appendix, we proceed to prove the statements related to the characterization of the optimal policy under the pricing and allocation mechanism defined in the paper. For notational ease, let  $\bar{V}$  be the unconditional value function before the realization of the demand takes place, i.e.,  $\bar{V}(\Gamma) = kV(H, \Gamma) + (1-k)V(L, \Gamma)$  for each  $\Gamma \in \Gamma$ . We also denote its partial derivative with respect to  $\Gamma$  by  $\bar{V}'$ , so  $\bar{V}'(\Gamma) = kV'(H, \Gamma) + (1-k)V'(L, \Gamma), \forall \Gamma \in \Gamma$ .

### A.1. Proof of Proposition 1

It is immediate to see that the Blackwell sufficiency conditions, i.e., monotonicity and discounting, are satisfied. This guarantees the existence of an optimal solution to Problem ( $\mathcal{P}$ ).

### A.2. Proof of Proposition 2

We consider a given state  $(D, \Gamma)$  and consider the optimal policy of Problem  $\mathcal{P}$ . For notational ease, we denote the policy by  $(p_r, p_t, p_l, q_r, q_t, q_l)$ , and removing the dependency on  $(D, \Gamma)$ . To prove each statement in sequence, we will reason by contradiction and construct an alternative solution  $(\tilde{p}_r, \tilde{p}_t, \tilde{p}_l, \tilde{q}_r, \tilde{q}_t, \tilde{q}_l)$  that strictly increases the platform's objective function.

*Proof that the incentive constraint  $\mathcal{IC}_r$  is binding:* Let us consider an optimal solution and assume by contradiction that the incentive constraint  $\mathcal{IC}_r$  is not binding, i.e.,  $q_r(1-p_r) > q_t[1-p_t]$ . Then we necessarily have  $p_r < 1$ , and we define an alternative solution by setting  $\tilde{p}_r = p_r + \varepsilon$ , with  $\varepsilon > 0$  such that  $\tilde{p}_r \leq 1$  and  $q_r(1-\tilde{p}_r) \geq q_t[1-p_t]$ . By construction, Constraints (3) and (7) are satisfied. Moreover, Constraints (8) and (9) remain unchanged. Finally, Constraint (4) is also satisfied as the right-hand side decreases with  $p_r$ . Therefore, the solution remains feasible. Moreover, we have  $q_r > 0$ , so the platform's value function strictly increases when  $p_r$  is increased to  $\tilde{p}_r$ . This contradicts the optimality of  $(p_r, p_t, p_l, q_r, q_t, q_l)$ .

*Proof that  $p_r \geq v_1$ :* Let us assume by contradiction that  $p_r < v_1$ .

If  $q_r > q_t$ , then we consider the deviation such that  $\tilde{p}_r = v_1$ ,  $\tilde{p}_t = v_1$ , and  $\tilde{p}_l = v_2$ , and  $\tilde{q}_r = q_r$ ,  $\tilde{q}_t = q_t$  and  $\tilde{q}_l = q_l$ . By construction, Construction, Constraint (7) is satisfied. Since  $\tilde{p}_r = v_1$ , Constraint (4) is also satisfied. Constraint (3) is also satisfied since  $\tilde{p}_r = \tilde{p}_t$  and  $q_r \geq q_t$ . As we did not change the allocation variables, Constraint (8) is also satisfied. Finally, the platform's objective function strictly increases, as  $\tilde{p}_r \tilde{q}_r \sigma D > p_r q_r \sigma D$ ,  $\tilde{q}_t \tilde{p}_t (1-\sigma)D \geq q_t p_t (1-\sigma)D$  and  $\tilde{q}_l \tilde{p}_l (1-\sigma)D \geq q_l p_l (1-\sigma)D$ , and the future value remains unchanged since the value of  $\tilde{\Gamma}' = (1-\tilde{q}_t)(1-\sigma)D = (1-q_t)(1-\sigma)D = \Gamma'$ . This contradicts with the optimality of  $(p_r, p_t, p_l, q_r, q_t, q_l)$ .

If  $0 < q_r < q_t$ , then we consider the deviation such that  $\tilde{p}_r = v_1$ ,  $\tilde{p}_t = v_1$ , and  $\tilde{p}_l = v_2$ ,  $\tilde{q}_r = \tilde{q}_t = sq_r + (1-\sigma)q_t$  and  $\tilde{q}_l = q_l$ . By construction, Construction, Constraint (7) is satisfied. Since  $\tilde{p}_r = \tilde{p}_t$  and  $\tilde{q}_r = \tilde{q}_t$ , Constraint (3) is satisfied and since  $\tilde{p}_r = v_1$ , Constraint (4) is satisfied. Moreover, by construction, we have  $\tilde{p}_r \tilde{q}_r \sigma D + \tilde{q}_t p_t (1-\sigma)D + \tilde{q}_l \Gamma = p_r q_r \sigma D + q_t p_t (1-\sigma)D + q_l \Gamma \leq 1$ , so Constraint (8) is satisfied. Finally, the platform's objective function strictly increases as follows:

$$\begin{aligned} & \tilde{p}_r \tilde{q}_r \sigma D + \tilde{p}_t \tilde{q}_t (1-\sigma)D + \tilde{p}_l \tilde{q}_l \Gamma + \delta \bar{V}(\tilde{\Gamma}') \\ &= v_1(\sigma q_r + (1-\sigma)q_t)\sigma D + v_1(\sigma q_r + (1-\sigma)q_t)(1-\sigma)D + v_2 q_l \Gamma + \delta \bar{V}((1-\tilde{q}_t)(1-\sigma)D) \end{aligned}$$

$$\begin{aligned}
&= \underbrace{q_r v_1 \sigma D}_{> q_r p_r \sigma D} + \underbrace{q_t v_1 (1 - \sigma) D}_{\geq q_t p_t (1 - \sigma) D} + \underbrace{v_2 q_t \Gamma}_{\geq p_t q_t \Gamma} + \delta \underbrace{\bar{V}((1 - \tilde{q}_t)(1 - \sigma) D)}_{\geq \bar{V}((1 - q_t)(1 - \sigma) D)} \\
&> p_r q_r \sigma D + p_t q_t (1 - \sigma) D + p_t q_t \Gamma + \delta \bar{V}(\Gamma')
\end{aligned}$$

If  $q_r = 0$ , then in order not to violate  $\mathcal{IC}_r$  we must also have  $p_t = 0$ . In this case, the platform could simply deviate to an  $\tilde{q}_r = 1$ ,  $\tilde{p}_r = 1$ ,  $\tilde{q}_t = q_t = 0$ ,  $\tilde{q}_l = \frac{q_l \Gamma - \sigma D}{\Gamma}$ ,  $\tilde{p}_l = v_2$ , and can get strictly better off by not violating any of the constraints.<sup>15</sup>

Note that the last inequality stems from the fact that  $\tilde{q}_t = s q_r + (1 - \sigma) q_t > q_t$  and  $V(D, \Gamma)$  is non-decreasing in  $\Gamma$ , which can be easily verified. This again contradicts with the optimality of  $(p_r, p_t, p_l, q_r, q_t, q_l)$ . Notice that this also proves that  $q_r \geq q_t$ .

*Proof that  $p_t = v_1$  and  $p_l = v_2$ :* Let us assume by contradiction that  $p_t < v_1$  (resp.  $p_l < v_2$ ). From the results above, we restrict our attention to the case where the incentive constraint  $\mathcal{IC}_r$  is binding and  $p_r \geq v_1$ . In the case where  $q_t = 0$  (resp.  $q_l = 0$ ), we can set  $p_t = v_1$  (resp.  $p_l = v_2$ ) without loss of generality. We now assume that  $q_t > 0$  (resp.  $q_l > 0$ ). Since  $p_t < v_1$  (resp.  $p_l < v_2$ ) and  $p_r \geq v_1$ , Constraint  $\mathcal{IC}_n$  is not binding. We consider  $\tilde{p}_t = p_t + \varepsilon$  (resp.  $\tilde{p}_l = p_l + \varepsilon$ ), with  $\varepsilon > 0$  such that  $p_t + \varepsilon \leq v_1$  (resp.  $p_l + \varepsilon \leq v_2$ ) and such that the incentive constraint  $\mathcal{IC}_n$  remains satisfied. By construction, this new solution does not violate Constraints (4) and (7). Constraints (8) and (9) remain unchanged, while Constraint (3) is loosened. Moreover, the new solution strictly increases the value function (Equation (5)). This contradicts the optimality of  $(p_r, p_t, p_l, q_r, q_t, q_l)$ .

*Proof that  $q_r = 1$ :* Suppose  $q_r < 1$  to get a contradiction. If the resource constraint (Constraint (8)) is not binding, then the platform could increase  $q_r$  by an arbitrarily small  $\varepsilon$  and can get strictly better off. This contradicts with the optimality.

If, on the other hand, the resource constraint (Constraint (8)) is binding, then  $q_r \sigma D + q_t (1 - \sigma) D + q_l \Gamma = 1$ . In the earlier steps we have already shown that  $p_t = v_1$ ,  $p_l = v_2$ , and (due to the binding  $\mathcal{IC}_r$ )  $p_r = 1 - \frac{q_t}{q_r} (1 - v_1) \geq v_1$ .

- If  $q_l > 0$ , then consider the following deviation,  $\tilde{q}_r = q_r + \varepsilon$ , and  $\tilde{q}_l = q_l - \varepsilon \frac{\sigma D}{\Gamma}$  without altering any other variable (we obviously have  $\Gamma > 0$  since  $q_l > 0$ ). Clearly, under this deviation all the constraints of problem  $\mathcal{P}$  are satisfied. Moreover the platform's objective value strictly increases as follows:

$$\begin{aligned}
&\tilde{p}_r \tilde{q}_r \sigma D + \tilde{q}_t p_t (1 - \sigma) D + \tilde{q}_l \tilde{p}_l \Gamma + \delta \bar{V}(\tilde{\Gamma}') \\
&= p_r q_r \sigma D + q_t p_t (1 - \sigma) D + q_l p_l \Gamma + \delta \bar{V}(\Gamma') + p_r \sigma D \varepsilon - p_l \sigma D \varepsilon \\
&> p_r q_r \sigma D + q_t p_t (1 - \sigma) D + q_l p_l \Gamma + \delta \bar{V}(\Gamma') \quad \text{because } p_r \geq v_1 > v_2 = p_l
\end{aligned}$$

This contradicts with the optimality of  $(p_r, p_t, p_l, q_r, q_t, q_l)$ .

- If  $q_l = 0$ , then we must have  $q_t > 0$  since the resource constraint (Constraints (8)) is binding, and we always have  $\sigma D < 1$  from Assumption 1. Consider the deviation in which  $\tilde{q}_r = q_r + \varepsilon$ ,  $\tilde{q}_t = q_t - \frac{\sigma}{1 - \sigma} \varepsilon$ , and  $\tilde{p}_r = 1 - \frac{\tilde{q}_t}{q_r} (1 - v_1)$ . By construction,  $\tilde{q}_r (1 - \tilde{p}_r) = \tilde{q}_t (1 - v_1)$ , so Constraint (3) is satisfied. Moreover, we have by construction  $\tilde{q}_r \sigma D + \tilde{q}_t (1 - \sigma) D + \tilde{q}_l \Gamma = q_r \sigma D + q_t (1 - \sigma) D + q_l \Gamma$ , so Constraint (8))

<sup>15</sup> In this case the value of  $p_t$  is irrelevant as there are no timely rides are allocated to price-sensitive agents.

is also satisfied. Finally, Constraint (4) is also satisfied, since it was satisfied under the original 6-tuple  $(p_r, p_t, p_l, q_r, q_t, q_l)$ , and now we have  $\tilde{p}_r > p_r$ . Note that the platform's objective function strictly increases as follows:

$$\begin{aligned}
& \tilde{p}_r \tilde{q}_r \sigma D + \tilde{p}_t \tilde{q}_t (1 - \sigma) D + \tilde{p}_l \tilde{q}_l \Gamma + \delta \bar{V}(\tilde{\Gamma}') \\
&= \left(1 - \frac{\tilde{q}_t}{\tilde{q}_r} (1 - v_1)\right) \tilde{q}_r \sigma D + v_1 \tilde{q}_t (1 - \sigma) D + p_l q_l \Gamma + \delta \bar{V}(\tilde{\Gamma}') \\
&= (\tilde{q}_r - \tilde{q}_t) \sigma D + \tilde{q}_t v_1 D + p_l q_l \Gamma + \delta \bar{V}(\tilde{\Gamma}') \\
&= (q_r - q_t) \sigma D + \left(\varepsilon + \frac{\sigma}{1 - \sigma} \varepsilon\right) \sigma D + q_t v_1 D - \frac{\sigma}{1 - \sigma} v_1 D \varepsilon + p_l q_l \Gamma + \delta \bar{V}(\tilde{\Gamma}') \\
&> (q_r - q_t) \sigma D + q_t v_1 D + p_l q_l \Gamma + \delta \bar{V}(\tilde{\Gamma}') \quad \text{because } v_1 < 1 \text{ and } \tilde{\Gamma}' > \Gamma' \\
&= p_r q_r \sigma D + p_t q_t (1 - \sigma) D + p_l q_l \Gamma + \delta \bar{V}(\tilde{\Gamma}') \quad \text{because Constraint (3) is binding}
\end{aligned}$$

This again contradicts with the optimality of  $(p_r, p_t, p_l, q_r, q_t, q_l)$ .

*Proof that If  $\Gamma > 0$ , then  $q_l(D, \Gamma) = \min\left\{1, \frac{1 - \sigma D - q_t(D, \Gamma)(1 - \sigma)D}{\Gamma}\right\}$ :* Let us assume that  $\Gamma > 0$  and, by contradiction, that  $q_l < 1$  and  $1 > q_r \sigma D + q_t(1 - \sigma)D + q_l \Gamma$ . Then we can define  $\tilde{q}_l = q_l + \varepsilon$ , with  $\varepsilon > 0$  such that  $\tilde{q}_l < 1$  and  $1 > q_r \sigma D + q_t(1 - \sigma)D + \tilde{q}_l \Gamma$ . By construction, Constraint (8) is satisfied, and Constraints (3), (7) and (9) remain unchanged. Moreover, note that, since we have proved that  $p_t = v_1$ ,  $p_l = v_2$  and  $p_r \geq v_1$ , Constraint (4) is automatically satisfied. Finally, since  $p_l = v_2 > 0$  and  $\Gamma > 0$ , the new solution strictly increases the value function. This contradicts the optimality of  $(p_r, p_t, p_l, q_r, q_t, q_l)$ .

*Proof that  $p_r = 1 - q_t(1 - v_1)$ :* We have already shown that the incentive constraint  $\mathcal{IC}_r$  is binding, i.e.,  $q_r(1 - p_r) = q_t[1 - p_t]$ . Since  $q_r = 1$  and  $p_t = v_1$ , it yields directly  $p_r = 1 - q_t(1 - v_1)$ .

This completes the proof.

### A.3. Proof of Proposition 3

Let us first re-write the Bellman equation for Problem (P):

$$\begin{aligned}
V(D, \Gamma) &= \max_{\Gamma' \in [\underline{\Gamma}_D, \bar{\Gamma}_D]} \tilde{V}(D, \Gamma, \Gamma') \\
&= \max_{\Gamma' \in [\underline{\Gamma}_D, \bar{\Gamma}_D]} \left\{ \sigma D \left(1 - \frac{(1 - \sigma)D - \Gamma'}{(1 - \sigma)D} (1 - v_1)\right) + ((1 - \sigma)D - \Gamma')v_1 + \min\{\Gamma, 1 - D + \Gamma'\}v_2 + \delta \bar{V}(\Gamma') \right\}
\end{aligned}$$

To see that the value function  $V(D, \Gamma)$  is non-decreasing in  $\Gamma$ , note that  $\sigma D \left(1 - \frac{(1 - \sigma)D - \Gamma'}{(1 - \sigma)D} (1 - v_1)\right) + ((1 - \sigma)D - \Gamma')v_1 + \min\{\Gamma, 1 - D + \Gamma'\}v_2$  is non-decreasing in  $\Gamma$ , and that the term  $\bar{V}(\Gamma')$  does not depend on  $\Gamma$ . Therefore, we directly obtain, for any  $D \in \{L, H\}$  and any  $\Gamma_a < \Gamma_b$ :

$$V(D, \Gamma_b) \geq \tilde{V}(D, \Gamma_b, \Gamma^*(D, \Gamma_a)) \geq \tilde{V}(D, \Gamma_a, \Gamma^*(D, \Gamma_a)) = V(D, \Gamma_a)$$

Let us now turn to the proof of concavity. We define  $\Gamma_\lambda = \lambda \Gamma_a + (1 - \lambda) \Gamma_b$ , and aim to show that  $V(D, \Gamma_\lambda) \geq \lambda V(D, \Gamma_a) + (1 - \lambda) V(D, \Gamma_b)$ . We proceed by value iteration. Specifically, we initialize  $V_0(D, \Gamma) = 0, \forall (D, \Gamma) \in \{H, L\} \times \mathbf{\Gamma}$ , and consider the following recursive sequence of functions, for  $n \geq 0$ :

$$\begin{aligned}
\bar{V}_{n-1}(\Gamma) &= k V_{n-1}(H, \Gamma) + (1 - k) V_{n-1}(L, \Gamma), \forall \Gamma \in \mathbf{\Gamma}, \\
\tilde{V}_n(D, \Gamma, \Gamma') &= \sigma D \left(1 - \frac{(1 - \sigma)D - \Gamma'}{(1 - \sigma)D} (1 - v_1)\right) + ((1 - \sigma)D - \Gamma')v_1 + \min\{\Gamma, 1 - D + \Gamma'\}v_2 + \delta \bar{V}_{n-1}(\Gamma'),
\end{aligned}$$

$$V_n(D, \Gamma) = \max_{\Gamma' \in [\Gamma_D, \bar{\Gamma}_D]} \tilde{V}_{n-1}(D, \Gamma, \Gamma').$$

By value iteration, we know that the sequence  $V_n$  converges to the optimal value function  $V$  (Bertsekas 2012). We will show by induction over  $n \geq 0$  that  $V_n(D, \Gamma_\lambda) \geq \lambda V_n(D, \Gamma_a) + (1 - \lambda)V_n(D, \Gamma_b)$ . By taking the limit when  $n \rightarrow \infty$ , this will yield concavity of  $V$ . First, note that the property is clearly satisfied for  $n = 0$ . We now assume that it holds for some  $n - 1$  and show it for  $n \geq 1$ .

For each  $\Gamma \in \Gamma$ , we introduce the following notations:

$$\Gamma_n^*(D, \Gamma) = \arg \max_{\Gamma' \in [\Gamma_D, \bar{\Gamma}_D]} \tilde{V}_n(D, \Gamma, \Gamma').$$

We also define  $y_\lambda = \lambda \Gamma_n^*(D, \Gamma_a) + (1 - \lambda)\Gamma_n^*(D, \Gamma_b)$ . We have:  $V_n(D, \Gamma_\lambda) \geq \tilde{V}_n(D, \Gamma_\lambda, y_\lambda)$ , which yields:

$$V_n(D, \Gamma_\lambda) \geq \sigma D \left( 1 - \frac{(1-\sigma)D - y_\lambda}{(1-\sigma)D} (1 - v_1) \right) + ((1-\sigma)D - y_\lambda)v_1 + \min\{\Gamma_\lambda, 1 - D + y_\lambda\}v_2 + \delta \bar{V}_{n-1}(y_\lambda)$$

Moreover, we have:

$$\begin{aligned} & \lambda V(D, \Gamma_a) + (1 - \lambda)V(D, \Gamma_b) \\ &= \lambda \left[ \sigma D \left( 1 - \frac{(1-\sigma)D - \Gamma_n^*(D, \Gamma_a)}{(1-\sigma)D} (1 - v_1) \right) + ((1-\sigma)D - \Gamma_n^*(D, \Gamma_a))v_1 + \min\{\Gamma_a, 1 - D + \Gamma_n^*(D, \Gamma_a)\}v_2 + \delta \bar{V}_{n-1}(\Gamma_n^*(D, \Gamma_a)) \right] \\ &+ (1 - \lambda) \left[ \sigma D \left( 1 - \frac{(1-\sigma)D - \Gamma_n^*(D, \Gamma_b)}{(1-\sigma)D} (1 - v_1) \right) + ((1-\sigma)D - \Gamma_n^*(D, \Gamma_b))v_1 + \min\{\Gamma_b, 1 - D + \Gamma_n^*(D, \Gamma_b)\}v_2 + \delta \bar{V}_{n-1}(\Gamma_n^*(D, \Gamma_b)) \right] \\ &= \sigma D \left( 1 - \frac{(1-\sigma)D - y_\lambda}{(1-\sigma)D} (1 - v_1) \right) + ((1-\sigma)D - y_\lambda)v_1 \\ &+ \lambda \min\{\Gamma_a, 1 - D + \Gamma_n^*(D, \Gamma_a)\}v_2 + (1 - \lambda) \min\{\Gamma_b, 1 - D + \Gamma_n^*(D, \Gamma_b)\}v_2 \\ &+ \delta [\lambda \bar{V}_{n-1}(\Gamma_n^*(D, \Gamma_a)) + (1 - \lambda)\bar{V}_{n-1}(\Gamma_n^*(D, \Gamma_b))] \end{aligned}$$

Note that the first line of the last equality stems from the fact that the first two terms of the function  $\tilde{V}_n$  are linear in  $\Gamma'$ . Moreover, from the induction hypothesis, we know that  $\bar{V}_{n-1}(y_\lambda) \geq \lambda \bar{V}_{n-1}(\Gamma_n^*(D, \Gamma_a)) + (1 - \lambda)\bar{V}_{n-1}(\Gamma_n^*(D, \Gamma_b))$ . Therefore, a sufficient condition is to show that:

$$\min\{\Gamma_\lambda, 1 - D + y_\lambda\} \geq \lambda \min\{\Gamma_a, 1 - D + \Gamma_n^*(D, \Gamma_a)\} + (1 - \lambda) \min\{\Gamma_b, 1 - D + \Gamma_n^*(D, \Gamma_b)\}$$

We separate four cases:

1. If  $\Gamma_a \leq 1 - D + \Gamma_n^*(D, \Gamma_a)$  and  $\Gamma_b \leq 1 - D + \Gamma_n^*(D, \Gamma_b)$ , then we have  $\Gamma_\lambda \leq 1 - D + y_\lambda$ , and we directly obtain:

$$\underbrace{\min\{\Gamma_\lambda, 1 - D + y_\lambda\}}_{=\Gamma_\lambda} = \lambda \underbrace{\min\{\Gamma_a, 1 - D + \Gamma_n^*(D, \Gamma_a)\}}_{=\Gamma_a} + (1 - \lambda) \underbrace{\min\{\Gamma_b, 1 - D + \Gamma_n^*(D, \Gamma_b)\}}_{=\Gamma_b}.$$

2. If  $\Gamma_a \leq 1 - D + \Gamma_n^*(D, \Gamma_a)$  and  $\Gamma_b > 1 - D + \Gamma_n^*(D, \Gamma_b)$ , then we distinguish two sub-cases:

- If  $\Gamma_\lambda \leq 1 - D + y_\lambda$ , we write  $\lambda \Gamma_a + (1 - \lambda)(1 - D + \Gamma_n^*(D, \Gamma_b)) < \lambda \Gamma_a + (1 - \lambda)\Gamma_b = \Gamma_\lambda$ , which yields:

$$\underbrace{\min\{\Gamma_\lambda, 1 - D + y_\lambda\}}_{=\Gamma_\lambda} \geq \lambda \underbrace{\min\{\Gamma_a, 1 - D + \Gamma_n^*(D, \Gamma_a)\}}_{=\Gamma_a} + (1 - \lambda) \underbrace{\min\{\Gamma_b, 1 - D + \Gamma_n^*(D, \Gamma_b)\}}_{=1 - D + \Gamma_n^*(D, \Gamma_b)}.$$

- If  $\Gamma_\lambda > 1 - D + y_\lambda$ , we write  $\lambda\Gamma_a + (1 - \lambda)(1 - D + \Gamma_n^*(D, \Gamma_b)) \leq \lambda(1 - D + \Gamma_n^*(D, \Gamma_a)) + (1 - \lambda)(1 - D + \Gamma_n^*(D, \Gamma_b)) = 1 - D + y_\lambda$ , which yields:

$$\underbrace{\min\{\Gamma_\lambda, 1 - D + y_\lambda\}}_{=1-D+y_\lambda} \geq \lambda \underbrace{\min\{\Gamma_a, 1 - D + \Gamma_n^*(D, \Gamma_a)\}}_{=\Gamma_a} + (1 - \lambda) \underbrace{\min\{\Gamma_b, 1 - D + \Gamma_n^*(D, \Gamma_b)\}}_{=1-D+\Gamma_n^*(D,\Gamma_b)}.$$

3. If  $\Gamma_a > 1 - D + \Gamma_n^*(D, \Gamma_a)$  and  $\Gamma_b \leq 1 - D + \Gamma_n^*(D, \Gamma_b)$ , we proceed as in Case 2. by symmetry.  
 4. If  $\Gamma_a > 1 - D + \Gamma_n^*(D, \Gamma_a)$  and  $\Gamma_b > 1 - D + \Gamma_n^*(D, \Gamma_b)$ , then we have  $\Gamma_\lambda > 1 - D + y_\lambda$ , and we directly obtain:

$$\underbrace{\min\{\Gamma_\lambda, 1 - D + y_\lambda\}}_{=1-D+y_\lambda} = \lambda \underbrace{\min\{\Gamma_a, 1 - D + \Gamma_n^*(D, \Gamma_a)\}}_{=1-D+\Gamma_n^*(D,\Gamma_a)} + (1 - \lambda) \underbrace{\min\{\Gamma_b, 1 - D + \Gamma_n^*(D, \Gamma_b)\}}_{=1-D+\Gamma_n^*(D,\Gamma_b)}.$$

This proves that  $V_n(D, \Gamma_\lambda) \geq \lambda V_n(D, \Gamma_a) + (1 - \lambda)V_n(D, \Gamma_b)$ , and completes the proof that the value function  $V$  is concave in  $\Gamma$ . We conclude by invoking the fact that a concave function is differentiable almost everywhere.

#### A.4. Some Remarks on the Partial Derivative of $V(D, \Gamma)$

In the remainder of the appendix, we extensively use the partial derivative of the value function  $V(D, \Gamma)$  with respect to  $\Gamma$ , denoted by  $V'(D, \Gamma)$ . From Equation (10), we have:

$$V(D, \Gamma) = \sigma D \left( 1 - \frac{(1 - \sigma)D - \Gamma^*(D, \Gamma)}{(1 - \sigma)D} (1 - v_1) \right) + ((1 - \sigma)D - \Gamma^*(D, \Gamma)) v_1 + \min\{\Gamma, 1 - D + \Gamma^*(D, \Gamma)\} v_2 + \bar{V}(\Gamma^*(D, \Gamma)).$$

Therefore we can get the derivative of  $V(D, \Gamma)$  as follows:

$$V'(D, \Gamma) = \frac{\partial \Gamma^*(D, \Gamma)}{\partial \Gamma} \left( \frac{\sigma - v_1}{1 - \sigma} + \bar{V}'(\Gamma^*(D, \Gamma)) \right) + \frac{\partial \min\{\Gamma, 1 - D + \Gamma^*(D, \Gamma)\}}{\partial \Gamma} v_2.$$

As we will see  $\frac{\partial \Gamma^*(D, \Gamma)}{\partial \Gamma}$  is either equal to 1 or 0. Moreover, the second term on the RHS can have two different values:  $v_2$  and 0. Therefore,  $V'(D, \Gamma)$  can take four different expressions:

$$\text{If } \frac{\partial \Gamma^*(D, \Gamma)}{\partial \Gamma} = 0 \text{ and } 1 - D + \Gamma^*(D, \Gamma) \leq \Gamma, \text{ then } V'(D, \Gamma) = 0 \quad (21)$$

$$\text{If } \frac{\partial \Gamma^*(D, \Gamma)}{\partial \Gamma} = 0 \text{ and } 1 - D + \Gamma^*(D, \Gamma) > \Gamma, \text{ then } V'(D, \Gamma) = v_2 \quad (22)$$

$$\text{If } \frac{\partial \Gamma^*(D, \Gamma)}{\partial \Gamma} = 1 \text{ and } 1 - D + \Gamma^*(D, \Gamma) \leq \Gamma, \text{ then } V'(D, \Gamma) = \frac{\sigma - v_1}{1 - \sigma} + \bar{V}'(\Gamma^*(D, \Gamma)) \quad (23)$$

$$\text{If } \frac{\partial \Gamma^*(D, \Gamma)}{\partial \Gamma} = 1 \text{ and } 1 - D + \Gamma^*(D, \Gamma) > \Gamma, \text{ then } V'(D, \Gamma) = \frac{\sigma - v_1}{1 - \sigma} + v_2 + \bar{V}'(\Gamma^*(D, \Gamma)) \quad (24)$$

Before proceeding further, we highlight the intuition behind each of these four cases:

- We have Equation (21), because in this case, increasing the value of  $\Gamma$  does not change the number of agents transferred to the next period, and the number of late rides.

- We have Equation (22), because in this case, increasing the value of  $\Gamma$  increases the number of late rides at a 1:1 rate, without altering the number of agents transferred to the next period.
- We have Equation (23), because in this case, increasing the value of  $\Gamma$ , increases the number of agents transferred to the next period at a 1:1 rate, without altering the number of late rides.
- We have Equation (24), because in this case, increasing the value of  $\Gamma$ , increases both the number of late rides, and the number of agents transferred to the next period at a 1:1 rate.

#### A.5. Proof of Lemma 1

It is sufficient to show that  $\frac{\partial \tilde{V}(D, \Gamma, \Gamma')}{\partial \Gamma'}$  is a weakly increasing function of  $\Gamma$  and  $D$ . Notice that the first two terms of the RHS of equation (11) are constant. The fourth term (i.e., the intertemporal effect) does not depend on the value of  $\Gamma$  and  $D$ . Therefore, we just need to show that the third term (i.e., the effect on late rides) is weakly increasing function of  $\Gamma$  and  $D$ . Notice that this term is a step function of  $\Gamma'$ , and satisfies:

$$\frac{\partial \min\{\Gamma, 1 - D + \Gamma'\} v_2}{\partial \Gamma'} = \begin{cases} v_2 & \text{if } \Gamma' < \Gamma + D - 1, \\ 0 & \text{if } \Gamma' \geq \Gamma + D - 1. \end{cases}$$

This directly shows that  $\frac{\partial \tilde{V}(D, \Gamma, \Gamma')}{\partial \Gamma'}$  is a weakly increasing function of  $\Gamma$  and  $D$ .

#### A.6. Proof of Lemma 2

Let us consider  $\Gamma_0$  such that  $\zeta^*(D, \Gamma_0) > \Gamma_0$ , i.e.,  $1 - D - \Gamma^*(D, \Gamma_0) > \Gamma_0$ . We have  $\frac{\partial \min\{\Gamma_0, 1 - D - \Gamma^*(D, \Gamma_0)\} v_2}{\partial \Gamma'} = 0$ . Since  $\tilde{V}$  is differentiable almost everywhere, we have from Equation (11):

$$\begin{aligned} \frac{\partial \tilde{V}(D, \Gamma_0, \Gamma^*(D, \Gamma_0) - \varepsilon)}{\partial \Gamma'} &= \frac{\sigma(1 - v_1)}{1 - \sigma} - v_1 + \delta \bar{V}'(\Gamma^*(D, \Gamma_0) - \varepsilon) \geq 0, \\ \frac{\partial \tilde{V}(D, \Gamma_0, \Gamma^*(D, \Gamma_0) + \varepsilon)}{\partial \Gamma'} &= \frac{\sigma(1 - v_1)}{1 - \sigma} - v_1 + \delta \bar{V}'(\Gamma^*(D, \Gamma_0) + \varepsilon) \leq 0. \end{aligned}$$

Let us now consider  $\Gamma \leq \Gamma_0$ . We have  $\zeta(D, \Gamma, \Gamma^*(D, \Gamma_0)) > \Gamma_0 \geq \Gamma$ , so  $\frac{\partial \min\{\Gamma, 1 - D + \Gamma^*(D, \Gamma_0)\} v_2}{\partial \Gamma'} = 0$ . We therefore obtain, as earlier:

$$\begin{aligned} \frac{\partial \tilde{V}(D, \Gamma, \Gamma^*(D, \Gamma_0) - \varepsilon)}{\partial \Gamma'} &= \frac{\sigma(1 - v_1)}{1 - \sigma} - v_1 + \delta \bar{V}'(\Gamma^*(D, \Gamma_0) - \varepsilon) \geq 0, \\ \frac{\partial \tilde{V}(D, \Gamma, \Gamma^*(D, \Gamma_0) + \varepsilon)}{\partial \Gamma'} &= \frac{\sigma(1 - v_1)}{1 - \sigma} - v_1 + \delta \bar{V}'(\Gamma^*(D, \Gamma_0) + \varepsilon) \leq 0. \end{aligned}$$

In other words, we have proved that:

$$\begin{aligned} \frac{\partial \tilde{V}(D, \Gamma, \Gamma^*(D, \Gamma_0) - \varepsilon)}{\partial \Gamma'} &= \frac{\partial \tilde{V}(D, \Gamma, \Gamma^*(D, \Gamma_0) - \varepsilon)}{\partial \Gamma'} && \geq 0 \\ \frac{\partial \tilde{V}(D, \Gamma, \Gamma^*(D, \Gamma_0) + \varepsilon)}{\partial \Gamma'} &= \frac{\partial \tilde{V}(D, \Gamma, \Gamma^*(D, \Gamma_0) + \varepsilon)}{\partial \Gamma'} && \leq 0 \end{aligned}$$

Therefore  $\Gamma^*(D, \Gamma_0)$  is also an optimal choice in state  $(D, \Gamma)$  too.<sup>16</sup>

#### A.7. Proof of Proposition 4

We have, from Equation (11):

$$\frac{\partial \tilde{V}(D, \Gamma, \Gamma')}{\partial \Gamma'} \geq \frac{\sigma(1-v_1)}{1-\sigma} - v_1 = \frac{\sigma-v_1}{1-\sigma} \geq 0, \quad \forall (D, \Gamma) \in \{H, L\} \times \Gamma, \quad \forall \Gamma' \in [\underline{\Gamma}_D, \bar{\Gamma}_D].$$

Therefore we always have:  $\Gamma^*(D, \Gamma) = \bar{\Gamma}_D$ ,  $\forall (D, \Gamma) \in \{H, L\} \times \Gamma$ . Notice that at the boundary  $v_1 = s$ ,  $\Gamma^*(D, \Gamma) = \bar{\Gamma}_D$  is not necessarily unique, but is still an optimal solution.

#### A.8. Proof of Lemma 3

By contradiction, let us consider  $\Gamma_0$  such that  $\zeta^*(D, \Gamma_0) < \Gamma_0$  and  $\zeta^*(D, \Gamma_0) < 1 - D + \bar{\Gamma}_D$  (i.e.,  $1 - D + \Gamma^*(D, \Gamma_0) < \Gamma_0$  and  $\Gamma^*(D, \Gamma) < \bar{\Gamma}_D$ ). This implies that:

$$\frac{\partial \min\{\Gamma_0, 1 - D + \Gamma^*(D, \Gamma_0)\}v_2}{\partial \Gamma'} = v_2.$$

In consequence:

$$\frac{\partial \tilde{V}(D, \Gamma_0, \Gamma^*(D, \Gamma_0))}{\partial \Gamma'} \geq \frac{\sigma(1-v_1)}{1-\sigma} - v_1 + v_2 > 0$$

This contradicts the fact that  $\Gamma^*(D, \Gamma_0)$  is optimal in state  $(D, \Gamma_0)$ . Notice that at the boundary  $v_1 = \sigma + (1 - \sigma)v_2$ , the claim is correct without loss of generality.

#### A.9. Proof of Lemma 4

Let us consider  $\Gamma_0$  such that  $\zeta^*(D, \Gamma_0) = \min\{\Gamma_0, 1 - D + \bar{\Gamma}_D\}$ . First, if  $\zeta^*(D, \Gamma_0) = 1 - D + \bar{\Gamma}_D$ , then  $\Gamma^*(D, \Gamma_0) = \bar{\Gamma}_D$  and from Lemma 1, we know that for each  $\Gamma \geq \Gamma_0$  we have  $\Gamma^*(D, \Gamma) = \bar{\Gamma}_D$ , so  $\zeta^*(D, \Gamma) = 1 - D + \bar{\Gamma}_D$ . We now assume that  $\Gamma^*(D, \Gamma_0) < \bar{\Gamma}_D$ . This implies that  $\zeta^*(D, \Gamma_0) = \Gamma_0$ , i.e.,  $\Gamma^*(D, \Gamma_0) = 1 - D + \Gamma_0$ . Therefore,  $\frac{\partial \min\{\Gamma_0, 1 - D + \Gamma^*(D, \Gamma_0)\}v_2}{\partial \Gamma'} = 0$ , and then, for an arbitrarily small  $\varepsilon > 0$ , we know that:

$$\frac{\partial \tilde{V}(D, \Gamma_0, \Gamma^*(D, \Gamma_0) + \varepsilon)}{\partial \Gamma'} = \frac{\sigma(1-v_1)}{1-\sigma} - v_1 + \delta \bar{V}'(\Gamma^*(D, \Gamma_0) + \varepsilon) \leq 0.$$

Let us consider  $\Gamma > \Gamma_0$  and assume by contradiction that  $\zeta^*(D, \Gamma) > \Gamma$ .<sup>17</sup> This can be re-written as  $1 - D + \Gamma^*(D, \Gamma) > \Gamma$ , or  $\Gamma^*(D, \Gamma) > D - 1 + \Gamma$ . This implies that, for some  $\varepsilon' > 0$  arbitrarily small, we have:

$$\frac{\partial \tilde{V}(D, \Gamma, D - 1 + \Gamma + \varepsilon')}{\partial \Gamma'} = \frac{\sigma(1-v_1)}{1-\sigma} - v_1 + \delta \bar{V}'(D - 1 + \Gamma + \varepsilon') > 0.$$

But for sufficiently small  $\varepsilon'$  and  $\varepsilon$  we must have  $D - 1 + \Gamma + \varepsilon' > \Gamma^*(D, \Gamma_0) + \varepsilon$ . Due to the concavity of the value function  $V$ , we therefore have  $\bar{V}'(D - 1 + \Gamma + \varepsilon') \leq \bar{V}'(\Gamma^*(D, \Gamma_0) + \varepsilon)$ . This is in contradiction with the two inequalities above.

<sup>16</sup> These arguments are based on the assumption that the value of  $\Gamma^*(D, \Gamma_0)$  is in the interior of the interval  $[\underline{\Gamma}_D, \bar{\Gamma}_D]$ , so that it is possible to take the partial derivative at  $\Gamma^*(D, \Gamma_0) \pm \varepsilon$ . However, the result goes through for boundary values as well, we just need to consider  $+\varepsilon$  or  $-\varepsilon$  for lower and upper bounds respectively.

<sup>17</sup> Notice that, in Lemma 3, we already argued that  $\zeta^*(D, \Gamma) \geq \min\{\Gamma, 1 - D + \bar{\Gamma}_D\}$ .

### A.10. Proof of Proposition 5

We already showed that, in region 2, the optimal policy function will be automatically pinned down once we figure out the value of  $\Gamma^*(D, 0)$ , as follows:

$$\Gamma^*(D, \Gamma) = \begin{cases} \Gamma^*(D, 0) & \text{if } \Gamma \leq 1 - D + \Gamma^*(D, 0), \\ \min\{\Gamma + D - 1, \bar{\Gamma}_D\} & \text{if } \Gamma \geq 1 - D + \Gamma^*(D, 0). \end{cases}$$

We therefore aim to determine  $\Gamma^*(D, 0)$  for  $D \in \{H, L\}$ . Note that  $\Gamma^*(D, 0)$  can be (i equal to the lower bound  $\underline{\Gamma}$  (ii an interior solution in the interval  $(\underline{\Gamma}_D, \bar{\Gamma}_D)$ , or (iii equal to the upper bound  $\bar{\Gamma}_D$

In the following, we list all these possibilities for each demand realization  $D \in \{H, L\}$  separately.

$$\text{H1) } \Gamma^*(H, 0) = \underline{\Gamma}_H$$

$$\text{L1) } \Gamma^*(L, 0) = \underline{\Gamma}_L$$

$$\text{H2) } \Gamma^*(H, 0) = \Gamma_1 \in (\underline{\Gamma}_H, \bar{\Gamma}_H)$$

$$\text{L2) } \Gamma^*(L, 0) = \Gamma_2 \in (\underline{\Gamma}_L, \bar{\Gamma}_L)$$

$$\text{H3) } \Gamma^*(H, 0) = \bar{\Gamma}_H$$

$$\text{L3) } \Gamma^*(L, 0) = \bar{\Gamma}_L$$

Now, we elaborate on all of these cases separately to find out whether and when these possibilities can be part of an optimal policy function  $\Gamma^*(D, \Gamma)$ . Before doing so, we first point out some initial observations regarding each of these cases. These observations will be useful in later stages. While describing underlying properties of these cases, the parameter value of  $\sigma$  becomes particularly important. In some cases, we further need to consider two subcases depending on whether  $\sigma < 0.5$  or  $\sigma \geq 0.5$ .

*Case H1:* If the optimal policy function satisfies H1, then we must have the following:

$$\Gamma^*(H, \Gamma) = \begin{cases} \Gamma + H - 1 & \text{if } \Gamma \leq 1 - \sigma H, \\ \bar{\Gamma}_H & \text{if } \Gamma \geq 1 - \sigma H. \end{cases}$$

$$\frac{\partial_+ \tilde{V}(H, 0, \underline{\Gamma}_H)}{\partial \Gamma'} = \frac{\sigma - v_1}{1 - \sigma} + \delta \bar{V}'_+(\underline{\Gamma}_H) \leq 0.$$

$$V'(H, \Gamma) = \begin{cases} \frac{\sigma - v_1}{1 - \sigma} + v_2 + \delta \bar{V}'(\Gamma + H - 1) & \text{if } \Gamma \leq 1 - \sigma H, & \text{(Equation (24))} \\ 0 & \text{if } \Gamma \geq 1 - \sigma H. & \text{(Equation (21))} \end{cases}$$

Notice that at  $\Gamma = 1 - \sigma H$ , the derivative  $V'(D, \Gamma)$  does not exist. The two expressions provided above, the first one is when  $\Gamma \leq 1 - \sigma H$ , and the other is when  $\Gamma \geq 1 - \sigma H$ , are respectively the left and the right derivatives of  $V(D, \Gamma)$  at  $\Gamma = 1 - \sigma H$ . In what follows, we follow this convention, in that the expressions provided at a kink point of a piece-wise defined derivative are respectively the left and the right derivatives at the kink point.

*Case H2:* If the optimal policy function satisfies H2, then we must have the following:

$$\Gamma^*(H, \Gamma) = \begin{cases} \Gamma_1 & \text{if } \Gamma \leq \Gamma_1 + 1 - H \\ \Gamma + H - 1 & \text{if } \Gamma \in [\Gamma_1 + 1 - H, 1 - \sigma H] \\ \bar{\Gamma}_H & \text{if } \Gamma \geq 1 - \sigma H \end{cases}$$

$$\frac{\partial_- \tilde{V}(H, 0, \Gamma_1)}{\partial \Gamma'} = \frac{\sigma - v_1}{1 - \sigma} + \delta \bar{V}'_-(\Gamma_1) \geq 0.$$

$$\frac{\partial_+ \tilde{V}(H, 0, \Gamma_1 \Gamma_1)}{\partial \Gamma'} = \frac{\sigma - v_1}{1 - \sigma} + \delta \bar{V}'_+(\Gamma_1) \leq 0.$$

$$V'(H, \Gamma) = \begin{cases} v_2 & \text{if } \Gamma \leq \Gamma_1 + 1 - H & \text{(Equation (22))} \\ \frac{\sigma - v_1}{1 - \sigma} + v_2 + \delta \bar{V}'(\Gamma + H - 1) & \text{if } \Gamma \in [\Gamma_1 + 1 - H, 1 - \sigma H] & \text{(Equation (24))} \\ 0 & \text{if } \Gamma \geq 1 - \sigma H & \text{(Equation (21))} \end{cases}$$

*Case H3:* If the optimal policy function satisfies H3, then we must have the following:

$$\Gamma^*(H, \Gamma) = \bar{\Gamma}_H, \quad \forall \Gamma \in \Gamma.$$

$$\frac{\partial_- \tilde{V}(H, 0, \bar{\Gamma}_H)}{\partial \Gamma'} = \frac{\sigma - v_1}{1 - \sigma} + \delta \bar{V}'_-(\bar{\Gamma}_H) \geq 0$$

$$V'(H, \Gamma) = \begin{cases} v_2 & \text{if } \Gamma \leq 1 - \sigma H & \text{(Equation (22))} \\ 0 & \text{if } \Gamma \geq 1 - \sigma H & \text{(Equation (21))} \end{cases}$$

*Case L1:* If the optimal policy function satisfies L1, then we must have the following:

$$\frac{\partial_+ \tilde{V}(L, 0, 0)}{\partial \Gamma'} = \frac{\sigma - v_1}{1 - \sigma} + \delta \bar{V}'_+(\Gamma_L) \leq 0.$$

*Subcase L1a:* If  $\sigma \geq 0.5$ , then

$$\Gamma^*(L, \Gamma) = \begin{cases} 0 & \text{if } \Gamma \leq 1 - L \\ \Gamma + L - 1 & \text{if } \Gamma \geq 1 - L \end{cases}$$

$$V'(L, \Gamma) = \begin{cases} v_2 & \text{if } \Gamma \leq 1 - L & \text{(Equation (22))} \\ \frac{\sigma - v_1}{1 - \sigma} + v_2 + \delta V'(\Gamma + L - 1) & \text{if } \Gamma \geq 1 - L & \text{(Equation (24))} \end{cases}$$

*Subcase L1b:* If  $\sigma < 0.5$ , then

$$\Gamma^*(L, \Gamma) = \begin{cases} 0 & \text{if } \Gamma \leq 1 - L \\ \Gamma + L - 1 & \text{if } 1 - L \leq \Gamma \leq 1 - \sigma L \\ \bar{\Gamma}_L & \text{if } \Gamma \geq 1 - \sigma L \end{cases}$$

$$V'(L, \Gamma) = \begin{cases} v_2 & \text{if } \Gamma \leq 1 - L & \text{(Equation (22))} \\ \frac{\sigma - v_1}{1 - \sigma} + v_2 + \delta V'(\Gamma + L - 1) & \text{if } 1 - L \leq \Gamma \leq 1 - \sigma L & \text{(Equation (24))} \\ 0 & \text{if } \Gamma \geq 1 - \sigma L & \text{(Equation (21))} \end{cases}$$

*Case L2:* If the optimal policy function satisfies L2, then we must have the following:

$$\begin{aligned}\frac{\partial_- \tilde{V}(L, 0, \Gamma_2)}{\partial \Gamma'} &= \frac{\sigma - v_1}{1 - \sigma} + \delta \bar{V}'_-(\Gamma_2) \geq 0. \\ \frac{\partial_+ \tilde{V}(L, 0, \Gamma_2)}{\partial \Gamma'} &= \frac{\sigma - v_1}{1 - \sigma} + \delta \bar{V}'_+(\Gamma_2) \leq 0.\end{aligned}$$

In this case we define subcases based on i) whether  $\sigma < 0.5$ , or  $\sigma \geq 0.5$ , and ii) whether  $\Gamma_2 < 1 - \sigma H$  or  $\Gamma_2 \geq 1 - \sigma H$ . Notice that when  $\Gamma_2 \geq 1 - \sigma H$ , we automatically have  $\sigma \geq 0.5$ . Therefore we have three subcases in total.

*Subcase L2a:* If  $\Gamma_2 < 1 - \sigma H$ , and  $\sigma \geq 0.5$ , then

$$\Gamma^*(L, \Gamma) = \begin{cases} \Gamma_2 & \text{if } \Gamma \leq 1 - L + \Gamma_2, \\ \Gamma + L - 1 & \text{if } \Gamma \geq 1 - L + \Gamma_2. \end{cases}$$

$$V'(L, \Gamma) = \begin{cases} v_2 & \text{if } \Gamma \leq 1 - L + \Gamma_2, & \text{(Equation (22))} \\ \frac{\sigma - v_1}{1 - \sigma} + v_2 + \delta V'(\Gamma + L - 1) & \text{if } \Gamma \geq 1 - L + \Gamma_2. & \text{(Equation (24))} \end{cases}$$

*Subcase L2a':* If  $\Gamma_2 < 1 - \sigma H$ , and  $\sigma < 0.5$ , then

$$\Gamma^*(L, \Gamma) = \begin{cases} \Gamma_2 & \text{if } \Gamma \leq 1 - L + \Gamma_2, \\ \Gamma + L - 1 & \text{if } \Gamma \in [1 - L + \Gamma_2, 1 - \sigma L], \\ \bar{\Gamma}_L & \text{if } \Gamma \geq 1 - \sigma L. \end{cases}$$

$$V'(L, \Gamma) = \begin{cases} v_2 & \text{if } \Gamma \leq 1 - L + \Gamma_2, & \text{(Equation (22))} \\ \frac{\sigma - v_1}{1 - \sigma} + v_2 + \delta V'(\Gamma + L - 1) & \text{if } \Gamma \in [1 - L + \Gamma_2, 1 - \sigma L], & \text{(Equation (24))} \\ 0 & \text{if } \Gamma \geq 1 - \sigma L. & \text{(Equation (21))} \end{cases}$$

*Subcase L2b:* If  $\Gamma_2 \geq 1 - \sigma H$ , then

$$\Gamma^*(L, \Gamma) = \Gamma_2, \quad \forall \Gamma \in \mathbf{\Gamma}.$$

$$V'(L, \Gamma) = v_2, \quad \forall \Gamma \in \mathbf{\Gamma}.$$

*Case L3:* If the optimal policy function satisfies L3, then we must have the following:

$$\Gamma^*(L, \Gamma) = \bar{\Gamma}_L, \quad \forall \Gamma \in \mathbf{\Gamma}.$$

$$\frac{\partial_- \tilde{V}(L, 0, \bar{\Gamma}_L)}{\partial \Gamma'} = \frac{\sigma - v_1}{1 - \sigma} + \delta \bar{V}'_-(\bar{\Gamma}_L) \geq 0$$

In this case, we have two subcases depending on the value of  $\sigma$ .

*Subcase L3a:* If  $\sigma \geq 0.5$ , then  $\bar{\Gamma}_L > 1 - \sigma H$  and:

$$V'(L, \Gamma) = v_2, \quad \forall \Gamma \in \mathbf{\Gamma}. \quad \text{(Equation (22))}$$

*Subcase L3b:* If  $\sigma < 0.5$ , then  $\bar{\Gamma}_L \leq 1 - \sigma H$  and:

$$V'(L, \Gamma) = \begin{cases} v_2 & \text{if } \Gamma \leq 1 - \sigma L, & \text{(Equation (22))} \\ 0 & \text{if } \Gamma \geq 1 - \sigma L. & \text{(Equation (21))} \end{cases}$$

We carry out an analysis by separately examining the cases that are based on the value of  $\Gamma^*(H, 0)$ , i.e., the cases H1 H2 and H3. Since the problem that we deal with is of a discrete nature, we are likely to end up with corner solutions. For the sake of a clarified exposition, as long as it does not cause any loss of generality, we focus on the corner solutions.

We first examine the instances in which we *must* have an interior solution for  $\Gamma^*(H, 0)$ , i.e., the case H2. Suppose  $\Gamma^*(H, 0) = \Gamma_1 \in (\underline{\Gamma}_H, \bar{\Gamma}_H)$ . The question is which one(s) of the cases L1, L2, and L3 may coexist with H2 in the optimal mechanism. The fact that we *must* have an interior solution for  $\Gamma^*(H, 0)$  implies that:

$$\frac{\sigma - v_1}{1 - \sigma} + \delta \bar{V}'(\underline{\Gamma}_H) > 0.$$

Therefore, we can immediately preclude the case L1. We can also preclude L2, because it would require that  $\Gamma(H, 0) = \Gamma_1 = \Gamma_2 = \Gamma(L, 0)$ ; and since  $\Gamma_1 > \underline{\Gamma}_H > 1 - \sigma H$ , this is possible only in subcase L2b. However in subcase L2b, we know that  $V'(L, \Gamma_1) = v_2$ . In addition, since  $\Gamma_1 \in (\underline{\Gamma}_H, \bar{\Gamma}_H)$ , we must have  $V'(H, \Gamma_1) = 0$  (Case H2). Therefore,  $\bar{V}'(\Gamma_1) = kV'(H, \Gamma_1) + (1 - k)V'(L, \Gamma_1) = (1 - k)v_2$ . Then in order to comply with the optimality conditions of H2, i.e.,  $\frac{\partial_- \bar{V}(H, 0, \Gamma_1)}{\partial \Gamma'} \geq 0$  and  $\frac{\partial_+ \bar{V}(H, 0, \Gamma_1)}{\partial \Gamma'} \leq 0$ , we must have  $v_1 = \sigma + \delta(1 - k)(1 - \sigma)v_2$ . But in this case, the platform is indifferent between setting  $\Gamma_1$  and  $\bar{\Gamma}_H$  in state  $(H, 0)$ , and simply can choose the former. Therefore we do not have to have an interior solution for  $\Gamma^*(H, 0)$ . An analogous contradiction follows in subcase L3a.

Thus, only Case L3b can co-exist with an interior solution for  $\Gamma(H, 0)$ . To comply with the optimality conditions of H2, we must have  $\Gamma_1 = 1 - \sigma L$ , since the value of  $V'(L, \Gamma)$  changes only at  $1 - \sigma L$ . Finally, since  $\bar{V}'(1 - \sigma L) = v_2$ , Condition H2 implies that  $\frac{\sigma - v_1}{1 - \sigma} + \delta(1 - k)v_2 \geq 0$ . All of these arguments regarding the case H2 is summarized in the following claim.

**CLAIM 1.** *In Region 2, if the optimal solution  $\Gamma(H, \Gamma)$  is unique and satisfies Case H2, then we have:*

- i)  $\Gamma_1 = 1 - \sigma L$ .
- ii)  $\sigma < 0.5$ , and  $\Gamma^*(L, \Gamma)$  satisfies L3b.
- iii)  $v_1 \leq \sigma + (1 - \sigma)\delta(1 - k)v_2$ .

Next, we focus on Case H3, i.e., the case in which we have  $\Gamma^*(H, 0) = \bar{\Gamma}_H$ . The underlying optimality condition in this case,  $\frac{\sigma - v_1}{1 - \sigma} + \delta \bar{V}'(\bar{\Gamma}_H) \geq 0$ . By concavity of  $\bar{V}$  (Proposition 3), this implies that  $\frac{\sigma - v_1}{1 - \sigma} + \delta \bar{V}'(\underline{\Gamma}_L) \geq \frac{\sigma - v_1}{1 - \sigma} + \delta \bar{V}'(\Gamma_2) \geq 0$ . This excludes Cases L1 and L2 and we only left with Case L3. Let us assume that Case H3 co-exists with Case L3b. However, we know that

$V'(H, \bar{\Gamma}_H) = 0$  (Case H3), and also  $V'(L, \bar{\Gamma}_H) = 0$  (Case L3b). Therefore,  $\frac{\sigma - v_1}{1 - \sigma} + \delta \bar{V}'_-(\bar{\Gamma}_H) = \frac{\sigma - v_1}{1 - \sigma} < 0$ , where the strict inequality follows from the fact that we are in Region 2 ( $v_1 > s$ ). This contradicts the above condition; hence, Case H3 can only co-exist with Case L3a. We then have  $\bar{V}'_-(\bar{\Gamma}_H) = v_2$ , so  $\frac{\sigma - v_1}{1 - \sigma} + \delta \bar{V}'_-(\bar{\Gamma}_H) \geq 0$  implies  $v_1 \leq \sigma + (1 - \sigma)\delta(1 - k)v_2$ . This is summarized in the following claim.

CLAIM 2. *In region 2, if the optimal solution of  $\Gamma(H, \Gamma)$  satisfies Case H3, then we have:*

i)  $\sigma \geq 0.5$  and  $\Gamma^*(L, \Gamma)$  satisfies L3a.

ii)  $v_1 \leq \sigma + (1 - \sigma)\delta(1 - k)v_2$ .

Finally, we turn to Case H1, i.e.,  $\Gamma^*(H, 0) = \underline{\Gamma}_H$ . Following the same strategy as before, we consider all the cases L1, L2, and L3, and see which one(s) can coexist with H1.

Let us first consider the coexistence of Case L3 with Case H1. Starting with Subcase L3a, we know that  $V'(L, \Gamma) = v_2$ , for each  $\Gamma \in \Gamma$ . Also, from Case H1, we know that  $V'(H, \underline{\Gamma}_H) = 0$ . Therefore,  $\frac{\partial \tilde{V}(H, 0, \Gamma')}{\partial \Gamma'} = \frac{\sigma - v_1}{1 - \sigma} + \delta(1 - k)v_2 \leq 0$  constant along  $\Gamma' \in [\underline{\Gamma}_H, \bar{\Gamma}_H]$ . Moreover, in Subcase L3a, we have  $\bar{\Gamma}_L \geq 1 - \sigma H$ , therefore Assumption 2 implies that  $V'(H, \bar{\Gamma}_L) = 0$ . Therefore, we must have:  $\frac{\partial_- \tilde{V}(L, 0, \bar{\Gamma}_L)}{\partial \Gamma'} = \frac{\sigma - v_1}{1 - \sigma} + \delta(1 - k)v_2 \geq 0$ . This together with the earlier expression requires that  $\frac{\sigma - v_1}{1 - \sigma} + \delta(1 - k)v_2 = 0$ . Hence,  $\frac{\partial \tilde{V}(H, 0, \Gamma')}{\partial \Gamma'} = 0$ , and the platform is indifferent between choosing anything in the interval  $[\underline{\Gamma}_H, \bar{\Gamma}_H]$  at  $(H, 0)$ . This possibility, however, is already covered in Case H3 (Claim 2).

Now consider the coexistence of Subcase L3b together with Case H1, for which a necessary condition is  $\bar{\Gamma}_L \leq \underline{\Gamma}_H$ . This condition, however, is automatically satisfied since  $\sigma < 0.5$ . Moreover, we have the following optimality conditions.

$$\begin{aligned} \frac{\partial_+ \tilde{V}(H, 0, \underline{\Gamma}_H)}{\partial \Gamma'} &= \frac{\sigma - v_1}{1 - \sigma} + \delta \bar{V}'_+(\underline{\Gamma}_H) \leq 0. \\ \frac{\partial_- \tilde{V}(L, 0, \bar{\Gamma}_L)}{\partial \Gamma'} &= \frac{\sigma - v_1}{1 - \sigma} + \delta \bar{V}'_-(\bar{\Gamma}_L) \geq 0. \end{aligned}$$

From Case H1, we know that  $V'(H, \underline{\Gamma}_H) = 0$ . In addition, since  $\bar{\Gamma}_L < \underline{\Gamma}_H < 1 - \sigma L$ , we have  $V'(L, \underline{\Gamma}_H) = V'(L, \bar{\Gamma}_L) = v_2$  (Case L3b). Finally, as we have  $\bar{\Gamma}_L < 1 - \sigma H$  in Case L3b, the condition of Case H1 implies that  $V'_-(H, \bar{\Gamma}_L) = \frac{\sigma - v_1}{1 - \sigma} + v_2 + \delta(1 - k)v_2$ . We therefore obtain:

$$\begin{aligned} \frac{\partial_+ \tilde{V}(H, 0, \underline{\Gamma}_H)}{\partial \Gamma'} &= \frac{\sigma - v_1}{1 - \sigma} + \delta(1 - k)v_2 \leq 0. \\ \frac{\partial_- \tilde{V}(L, 0, \bar{\Gamma}_L)}{\partial \Gamma'} &= \frac{\sigma - v_1}{1 - \sigma} + \delta \left( k \left[ \frac{\sigma - v_1}{1 - \sigma} + v_2 + \delta(1 - k)v_2 \right] + (1 - k)v_2 \right) \geq 0. \end{aligned}$$

The following claim summarizes all of these arguments to characterize the conditions under which the optimal solution satisfies Cases H1 and L3b.

CLAIM 3. *In region 2, if the optimal solution  $\Gamma^*(H, \Gamma)$  satisfies Cases H1 and L3b together, then we must have:*

i)  $\sigma < 0.5$ .

ii)  $v_1 \in [\sigma + (1 - \sigma)\delta(1 - k)v_2, \sigma + (1 - \sigma)\frac{\delta + \delta^2(1 - k)k}{1 + \delta k}v_2]$ .

We now investigate the coexistence of Case H1 and Case L1. We then have:

$$\frac{\partial_+ \tilde{V}(L, 0, 0)}{\partial \Gamma'} = \frac{\sigma - v_1}{1 - \sigma} + \delta \bar{V}'_+(\Gamma_L) \leq 0.$$

We have  $V'(L, 0) = v_2$  (Case L1). Moreover, we obtain from the conditions of Case H1:

$$\begin{aligned} V'_+(H, 0) &= \frac{\sigma - v_1}{1 - \sigma} + v_2 + \delta k V'_+(H, H - 1) + \delta(1 - k)V'_+(L, H - 1) \\ &= \frac{\sigma - v_1}{1 - \sigma} + v_2 + \delta(1 - k) \left( \frac{\sigma - v_1}{1 - \sigma} + v_2 + \delta k V'_+(H, 0) + \delta(1 - k)v_2 \right) \end{aligned}$$

We obtain:

$$V'_+(H, 0) = \frac{\sigma - v_1}{1 - \sigma} \left( \frac{1 + \delta(1 - k)}{1 - \delta^2(1 - k)k} \right) + v_2 \left( \frac{1 + \delta(1 - k) + \delta^2(1 - k)^2}{1 - \delta^2(1 - k)k} \right)$$

Therefore the condition that is needed for the optimality of  $\Gamma_L$  in state  $(L, 0)$  translates into:  $v_1 \geq \sigma + (1 - \sigma)\frac{\delta + \delta^2(1 - k)k}{1 + \delta k}v_2$ . Then the next claim summarizes these arguments.

**CLAIM 4.** *In region 2, if the optimal solution  $\Gamma(H, \Gamma)$  satisfies Cases H1 and L1, then we must have:  $v_1 \geq \sigma + (1 - \sigma)\frac{\delta + \delta^2(1 - k)k}{1 + \delta k}v_2$ .*

We examine whether it is possible to have H1 and L2 in an optimal solution. First, consider Case L2b, i.e., when  $\Gamma_2 \geq 1 - \sigma H$ , which also requires that  $\sigma \geq 0.5$ . We know that  $1 - L + \Gamma_2 \geq 1 - L + 1 - \sigma H = (1 - \sigma)H$  per Assumption 2. Therefore,  $\Gamma \leq 1 - L + \Gamma_2$  for all  $\Gamma \in \mathbf{\Gamma}$ . Since we are in Region L2, this implies that  $V'(L, \Gamma) = v_2$  for all  $\Gamma \in \mathbf{\Gamma}$ . Therefore,

$$\begin{aligned} \frac{\partial \tilde{V}(L, 0, \Gamma')}{\partial \Gamma'} &= \frac{\sigma - v_1}{1 - \sigma} + \delta (kV'(H, \Gamma') + (1 - k)V'(L, \Gamma')) \\ &= \frac{\sigma - v_1}{1 - \sigma} + \delta(1 - k)v_2 \quad \text{because } V'(H, \Gamma) = 0 \text{ (Case H1) and } V'(L, \Gamma) = v_2 \end{aligned}$$

If  $\frac{\sigma - v_1}{1 - \sigma} + \delta(1 - k)v_2 > 0$ , then  $\Gamma_2$  would be suboptimal at  $(L, 0)$ , since the platform could simply choose  $\bar{\Gamma}_L$  and get strictly better off. Therefore, we must have  $\frac{\sigma - v_1}{1 - \sigma} + \delta(1 - k)v_2 \leq 0$ . If  $\frac{\sigma - v_1}{1 - \sigma} + \delta(1 - k)v_2 = 0$ , then all the values within the interval  $[1 - \sigma H, \bar{\Gamma}_L]$  are optimal in state  $(L, 0)$ . If on the other hand,  $\frac{\sigma - v_1}{1 - \sigma} + \delta(1 - k)v_2 < 0$ , then  $1 - \sigma H$  is strictly better than all the other values included in the interval  $[1 - \sigma H, \bar{\Gamma}_L]$  at state  $(L, 0)$ . In consequence,  $\Gamma_2 = 1 - \sigma H$  without loss of any generality. We further need to make sure that setting  $1 - \sigma H - \varepsilon$  is not strictly better at  $(L, 0)$ .

In other words, we must have:

$$\frac{\partial_- \tilde{V}(L, 0, \Gamma')}{\partial \Gamma'} = \frac{\sigma - v_1}{1 - \sigma} + \delta (kV'_-(H, \Gamma') + (1 - k)V'_-(L, \Gamma')) \geq 0.$$

But we know that:

$$V'_-(H, 1 - \sigma H) = \frac{\sigma - v_1}{1 - \sigma} + v_2 + \delta(1 - k)v_2.$$

Then by using this we can get the corresponding necessary condition. The following claim summarizes these findings.

CLAIM 5. *In region 2, if the optimal solution  $\Gamma(H, \Gamma)$  satisfies Cases H1 and L2b, then  $\Gamma_2 = 1 - \sigma H$  without loss of any generality. Moreover, we have:*

i)  $\sigma \geq 0.5$ .

ii)  $v_1 \in [\sigma + (1 - \sigma)\delta(1 - k)v_2, \sigma + (1 - \sigma)\frac{\delta + \delta^2(1 - k)k}{1 + \delta k}v_2]$ .

If we have Case H1 together with Case L2a, i.e.,  $\Gamma_2 < 1 - \sigma H$ , and  $\sigma \geq 0.5$ , then following some algebra one can show that:

$$V'(H, \Gamma) = \begin{cases} \frac{\sigma - v_1}{1 - \sigma} + v_2(1 + \delta(1 - k)) & \text{if } \Gamma \leq \Gamma_2 \\ \frac{\sigma - v_1}{1 - \sigma} \left( \frac{1 + \delta(1 - k)}{1 - \delta^2 k(1 - k)} \right) + v_2 \left( \frac{1 + \delta(1 - k) + \delta^2(1 - k)^2}{1 - \delta^2 k(1 - k)} \right) & \text{if } \Gamma \in [\Gamma_2, 1 - \sigma H] \\ 0 & \text{if } \Gamma \in [1 - \sigma H, (1 - \sigma)H] \end{cases}$$

$$V'(L, \Gamma) = \begin{cases} v_2 & \text{if } \Gamma \leq 1 - L + \Gamma_2 \\ \frac{\sigma - v_1}{1 - \sigma} \left( \frac{1 + \delta k}{1 - \delta^2 k(1 - k)} \right) + v_2 \left( \frac{1 + \delta}{1 - \delta^2 k(1 - k)} \right) & \text{if } \Gamma \geq 1 - L + \Gamma_2 \end{cases}$$

The optimality conditions can be written as follows:

$$\begin{aligned} \frac{\partial_- \tilde{V}(L, 0, \Gamma_2)}{\partial \Gamma'} &= \frac{\sigma - v_1}{1 - \sigma} + \delta \bar{V}'_-(\Gamma_2) \geq 0. \\ \frac{\partial_+ \tilde{V}(L, 0, \Gamma_2)}{\partial \Gamma'} &= \frac{\sigma - v_1}{1 - \sigma} + \delta \bar{V}'_+(\Gamma_2) \leq 0. \end{aligned}$$

It yields:

$$\begin{aligned} \frac{\sigma - v_1}{1 - \sigma} + \delta k \left( \frac{\sigma - v_1}{1 - \sigma} + v_2(1 + \delta(1 - k)) \right) + \delta(1 - k)v_2 &\geq 0 \\ \frac{\sigma - v_1}{1 - \sigma} + \delta k \left( \frac{\sigma - v_1}{1 - \sigma} \left( \frac{1 + \delta(1 - k)}{1 - \delta^2 k(1 - k)} \right) + v_2 \left( \frac{1 + \delta(1 - k) + \delta^2(1 - k)^2}{1 - \delta^2 k(1 - k)} \right) \right) + \delta(1 - k)v_2 &\leq 0 \end{aligned}$$

Following some simple algebra, one can find that these inequalities respectively boil down to:

$$\begin{aligned} \frac{\sigma - v_1}{1 - \sigma} (1 + \delta k) + v_2(\delta + \delta^2(1 - k)k) &\geq 0. \\ \frac{\sigma - v_1}{1 - \sigma} (1 + \delta k) + v_2(\delta + \delta^2(1 - k)k) &\leq 0. \end{aligned}$$

These two conditions are satisfied together if and only if

$$\frac{\sigma - v_1}{1 - \sigma} (1 + \delta k) + v_2(\delta + \delta^2(1 - k)k) = 0.$$

We obtain, for each  $\Gamma' \in (\Gamma_2, 1 - \sigma H]$ :

$$\begin{aligned} \frac{\partial \tilde{V}(L, 0, \Gamma')}{\partial \Gamma'} &= \frac{\sigma - v_1}{1 - \sigma} + \delta \left( \underbrace{k V'(H, \Gamma')}_{=V'(H, \Gamma_2)} + (1 - k) \underbrace{V'(L, \Gamma')}_{=V'(L, \Gamma_2)} \right) \\ &= \frac{\sigma - v_1}{1 - \sigma} + \delta k \left[ \frac{\sigma - v_1}{1 - \sigma} \left( \frac{1 + \delta(1 - k)}{1 - \delta^2 k(1 - k)} \right) + v_2 \left( \frac{1 + \delta(1 - k) + \delta^2(1 - k)^2}{1 - \delta^2 k(1 - k)} \right) \right] + \delta(1 - k)v_2 \\ &= 0 \end{aligned}$$

The equality stems from the fact that  $V'(H, \Gamma')$ , and  $V'(L, \Gamma')$  stays constant over  $\Gamma' \in (\Gamma_2, 1 - \sigma H]$ . Therefore, the platform does not get hurt by increasing  $\Gamma^*(L, 0)$  up to  $1 - \sigma H$ . Hence it is without loss of generality to assume that  $\Gamma_2 = 1 - \sigma H$ , and this case is already covered in Case L2b (Claim 5).

If we have Case H1 together with Case L2a', i.e.,  $\Gamma_2 < 1 - \sigma H$ , and  $\sigma < 0.5$ , then following some algebra one can show that:

$$V'(H, \Gamma) = \begin{cases} \frac{\sigma - v_1}{1 - \sigma} + v_2(1 + \delta(1 - k)) & \text{if } \Gamma \leq \Gamma_2 \\ \frac{\sigma - v_1}{1 - \sigma} \left( \frac{1 + \delta(1 - k)}{1 - \delta^2 k(1 - k)} \right) + v_2 \left( \frac{1 + \delta(1 - k) + \delta^2(1 - k)^2}{1 - \delta^2 k(1 - k)} \right) & \text{if } \Gamma \in [\Gamma_2, (1 - \sigma)L] \\ \frac{\sigma - v_1}{1 - \sigma} + v_2 & \text{if } \Gamma \in [(1 - \sigma)L, 1 - \sigma H] \\ 0 & \text{if } \Gamma \in [1 - \sigma H, (1 - \sigma)H] \end{cases}$$

$$V'(L, \Gamma) = \begin{cases} v_2 & \text{if } \Gamma \leq 1 - L + \Gamma_2 \\ \frac{\sigma - v_1}{1 - \sigma} \left( \frac{1 + \delta k}{1 - \delta^2 k(1 - k)} \right) + v_2 \left( \frac{1 + \delta}{1 - \delta^2 k(1 - k)} \right) & \text{if } \Gamma \in [1 - L + \Gamma_2, 1 - \sigma L] \\ 0 & \text{if } \Gamma \in [1 - \sigma L, (1 - \sigma)H] \end{cases}$$

Then checking the conditions that guarantee the optimality of  $\Gamma_2$  at  $(L, 0)$ , we get the same expressions above when we analyzed coexistence of Case L2a and Case H1. More precisely, putting the optimality conditions together, we reach that, it is possible to sustain Case H1 together with Case L2a' in an optimal policy, only if the following condition holds.

$$\frac{\sigma - v_1}{1 - \sigma} (1 + \delta k) + v_2 (\delta + \delta^2(1 - k)k) = 0.$$

This in turn implies that:

$$\frac{\partial \tilde{V}(L, 0, \Gamma')}{\partial \Gamma'} = \frac{\sigma - v_1}{1 - \sigma} + \delta (k V'_-(H, \Gamma') + (1 - k) V'_-(L, \Gamma')) = 0, \quad \forall \Gamma' \in (\Gamma_2, (1 - \sigma)L].$$

Then the platform is indifferent between setting anything in  $(\Gamma_2, (1 - \sigma)L]$  in state  $(L, 0)$ . Then this possibility is already covered in the coexistence of Cases H1 and L3b (Claim 3).

We conclude by putting Claims 1, 2, 3, 5, and 4 together.

### A.11. Proof of Lemma 5

Let us consider a state  $(D, \Gamma_0)$  such that  $\zeta^*(D, \Gamma_0) < \Gamma_0$ , i.e., the number of drivers that the platform can assign a late ride in state  $(D, \Gamma_0)$  is lower than the number of agents waiting for a late ride. Thus transferring more people to the next period in this optimal outcome would also increase the number of late rides that the platform generates in this period. However, it is sub-optimal to do so. Therefore, for  $\varepsilon > 0$  sufficiently small, we obtain from Equation (11), using the fact that  $\frac{\partial \min\{\Gamma_0, 1-D-\Gamma^*(D, \Gamma_0)\}v_2}{\partial \Gamma'} = v_2$ :

$$\frac{\partial V(D, \Gamma_0, \Gamma^*(D, \Gamma_0) + \varepsilon)}{\partial \Gamma'} = \frac{\sigma - v_1}{1 - \sigma} + v_2 + \delta \bar{V}'(\Gamma^*(D, \Gamma_0) + \varepsilon) \leq 0$$

Moreover, for each  $\Gamma \geq \Gamma_0$ , we have  $\frac{\partial \min\{\Gamma_0, 1-D-\Gamma^*(D, \Gamma_0)\}v_2}{\partial \Gamma'} = v_2$ , so for  $\varepsilon > 0$  sufficiently small:

$$\begin{aligned} \frac{\partial V(D, \Gamma, \Gamma^*(D, \Gamma_0) + \varepsilon)}{\partial \Gamma'} &= \frac{\sigma - v_1}{1 - \sigma} + v_2 + \delta \bar{V}'(\Gamma^*(D, \Gamma_0) + \varepsilon) \\ &= \frac{\partial V(D, \Gamma_0, \Gamma^*(D, \Gamma_0) + \varepsilon)}{\partial \Gamma'} \\ &\leq 0 \end{aligned}$$

Therefore, at each state  $(D, \Gamma)$  satisfying  $\Gamma \geq \Gamma_0$ , we have  $\Gamma^*(D, \Gamma) \leq \Gamma^*(D, \Gamma_0)$ . At the same time, we know from Lemma 1 that  $\Gamma^*(D, \Gamma) \geq \Gamma^*(D, \Gamma_0)$ . This concludes the proof.

### A.12. Proof of Lemma 6

In order to prove Lemma 6 we first show that  $V'(D, \Gamma) \leq v_2$  for each  $D \in \{H, L\}$ . And for this we consider the two cases  $D = L$ , and  $D = H$  separately.

- Let us prove that  $V'(L, \Gamma) \leq v_2$ . Note, first, that  $V'(L, 0) = v_2$ . Indeed, from Lemma 2, we already know that  $\Gamma^*(L, \Gamma)$  is constant over  $[0, 1 - L]$  because  $\zeta^*(L, \Gamma) = 1 - L + \Gamma^*(L, \Gamma) > \Gamma$  for each  $\Gamma < 1 - L$ . Therefore  $\frac{\partial \Gamma^*(L, \Gamma)}{\partial \Gamma} = 0$ , and since  $1 - L + \Gamma^*(L, \Gamma) > \Gamma$  when  $\Gamma \in [0, 1 - L]$ , we have  $V'(L, 0) = v_2$  (Equation (22)). Given the concavity of  $V(D, \Gamma)$  as a function of  $\Gamma$  (Proposition 3), we obtain  $V'(L, \Gamma) \leq v_2$  for each  $\Gamma \in \mathbf{\Gamma}$ .
- We now prove that  $V'(H, \Gamma) \leq v_2$ . We consider two possibilities separately: 1)  $\Gamma^*(H, 0) > \underline{\Gamma}_H$ , and 2)  $\Gamma^*(H, 0) = \underline{\Gamma}_H$ .
  - If  $\Gamma^*(H, 0) > \underline{\Gamma}_H$ , then  $\zeta^*(H, 0) > 0$ . This implies that  $\Gamma^*(H, \Gamma) = \Gamma^*(H, 0)$  for all  $\Gamma \in [0, \zeta^*(H, 0))$  (Lemma 2). In other words, we have  $\frac{\partial \Gamma^*(H, \Gamma)}{\partial \Gamma} = 0$ , and  $1 - H + \Gamma^*(H, \Gamma) > \Gamma$  for all  $\Gamma \in [0, \zeta^*(H, 0))$ . Then from Equation (22), we must have  $V'(H, 0) = v_2$ . Then  $V'(H, \Gamma) \leq v_2$  for all  $\Gamma \in \mathbf{\Gamma}$  per concavity.
  - We now treat the case where  $\Gamma^*(H, 0) = \underline{\Gamma}_H$ .
    - If  $\Gamma^*(H, \Gamma) = \underline{\Gamma}_H = H - 1$  for all  $\Gamma \in \mathbf{\Gamma}$ . We have  $\frac{\partial \Gamma^*(H, \Gamma)}{\partial \Gamma} = 0$ . Moreover,  $1 - H + \Gamma^*(H, \Gamma) \leq 1 - H + (1 - \sigma)H = 1 - \sigma H$  and  $\Gamma = H - 1 \geq 1 - \sigma H$  per Assumption 2. Therefore,  $\frac{\partial \min\{\Gamma, 1-H+\Gamma^*(H, \Gamma)\}}{\partial \Gamma} = 0$  and  $V'(H, \Gamma) = 0$  for all  $\Gamma \in \mathbf{\Gamma}$ .

If  $\Gamma^*(H, \Gamma)$  is not constant over  $\Gamma$ , then we prove that there exists  $\bar{\varepsilon} > 0$  such that  $\Gamma^*(H, \varepsilon) = \underline{\Gamma}_H + \varepsilon$  for all  $\varepsilon < \bar{\varepsilon}$ . Indeed, let us assume that  $\Gamma^*(H, \varepsilon) > \underline{\Gamma}_H + \varepsilon$ . Then  $\zeta^*(H, \varepsilon) > \varepsilon$  and, from Lemma 2,  $\Gamma^*(H, 0) = \Gamma^*(H, \varepsilon)$ . This contradicts that  $\Gamma^*(H, 0) = \underline{\Gamma}_H$ . Conversely, if  $\Gamma^*(H, \varepsilon) < \underline{\Gamma}_H + \varepsilon$  for every  $\varepsilon > 0$ , then  $\zeta^*(H, \varepsilon) < \varepsilon$  so from Lemma 5, we need to have  $\Gamma^*(H, \Gamma) = \Gamma^*(H, 0)$  for all  $\Gamma$ . This contradicts the fact that  $\Gamma^*(H, \Gamma)$  is not constant over  $\Gamma$ .

Then we have  $\frac{\partial \Gamma^*(H, \Gamma)}{\partial \Gamma} = 1$  so per Equation (24):

$$V'(H, 0) = \frac{\sigma - v_1}{1 - \sigma} + v_2 + \delta \bar{V}'(H, H - 1)$$

We now prove that  $V'(H, H - 1) = 0$ . First, note that:  $1 - H + \Gamma^*(H, \Gamma) > \Gamma$  for each  $\Gamma \geq H - 1$ , since  $1 - H - \bar{\Gamma}_H = 1 - \sigma H < H - 1$  by Assumption 2m so we have  $\frac{\partial \min\{\Gamma, 1 - H + \Gamma^*(H, \Gamma)\}}{\partial \Gamma} = 0$ . We now show that  $\Gamma^*(H, \Gamma)$  stays constant for  $\Gamma \geq H - 1$ . By contradiction, suppose  $\Gamma^*(H, \Gamma)$  does not stay constant for  $\Gamma \geq H - 1$ . Then by monotonicity (Lemma 1), there exist  $\Gamma_0$  and  $\Gamma_1$  satisfying  $H - 1 \leq \Gamma_0 < \Gamma_1$ , such that  $\Gamma^*(H, \Gamma_1) > \Gamma^*(H, \Gamma_0)$ , and  $\Gamma^*(H, \Gamma_0)$  is not optimal at  $(H, \Gamma_1)$ . Therefore:

$$\begin{aligned} V(H, \Gamma_1) &= \sigma H \left( 1 - \frac{(1-\sigma)H - \Gamma^*(H, \Gamma_1)}{(1-\sigma)H} (1 - v_1) \right) + ((1 - \sigma)H - \Gamma^*(H, \Gamma_1))v_1 + \\ &\quad (1 - H + \Gamma^*(H, \Gamma_1)) + \delta \bar{V}(\Gamma^*(H, \Gamma_1)) \\ &> \sigma H \left( 1 - \frac{(1-\sigma)H - \Gamma^*(H, \Gamma_0)}{(1-\sigma)H} (1 - v_1) \right) + ((1 - \sigma)H - \Gamma^*(H, \Gamma_0))v_1 + \\ &\quad (1 - H + \Gamma^*(H, \Gamma_0)) + \delta \bar{V}(\Gamma^*(H, \Gamma_0)) \end{aligned}$$

But from the optimality of  $\Gamma^*(H, \Gamma_0)$  in state  $(H, \Gamma_0)$ , we know that:

$$\begin{aligned} V(H, \Gamma_0) &= \sigma H \left( 1 - \frac{(1-\sigma)H - \Gamma^*(H, \Gamma_0)}{(1-\sigma)H} (1 - v_1) \right) + \\ &\quad ((1 - \sigma)H - \Gamma^*(H, \Gamma_0))v_1 + (1 - H + \Gamma^*(H, \Gamma_0)) + \delta \bar{V}(\Gamma^*(H, \Gamma_0)) \\ &\geq \sigma H \left( 1 - \frac{(1-\sigma)H - \Gamma^*(H, \Gamma_1)}{(1-\sigma)H} (1 - v_1) \right) + \\ &\quad ((1 - \sigma)H - \Gamma^*(H, \Gamma_1))v_1 + (1 - H + \Gamma^*(H, \Gamma_1)) + \delta \bar{V}(\Gamma^*(H, \Gamma_1)) \end{aligned}$$

This gives us a contradiction. Therefore,  $\Gamma^*(H, \Gamma)$  stays constant for  $\Gamma \geq H - 1$  and  $\frac{\partial \Gamma^*(H, \Gamma)}{\partial \Gamma} = 0$ . It comes  $V'(H, H - 1) = 0$ . We also showed that  $V'(L, H - 1) \leq v_2$ . Finally we have  $\frac{\sigma - v_1}{1 - \sigma} + v_2 < 0$  since we are in Region 3. Therefore,  $V'(H, 0) \leq 0 + \delta \times (k \times 0 + (1 - k) \times v_2) = v_2$ , and hence by concavity  $V'(H, \Gamma) \leq v_2$ , for each  $\Gamma \in \mathbf{\Gamma}$ .

In summary, we have shown that  $V'(D, \Gamma) \leq v_2$  for each  $(D, \Gamma) \in \{H, L\} \times \mathbf{\Gamma}$ . We now use that result to show that  $\Gamma^*(H, 0) = \underline{\Gamma}_D$ , for each  $D \in \{H, L\}$ . Suppose by contradiction that  $\Gamma^*(D, 0) > \underline{\Gamma}_D$  for some  $D \in \{H, L\}$ . If  $\Gamma^*(H, 0) > \underline{\Gamma}_H$ , then there exists an arbitrarily small  $\varepsilon > 0$  such that:

$$\frac{\partial \tilde{V}(H, 0, \Gamma^*(H, 0) - \varepsilon)}{\partial \Gamma} = \frac{\sigma - v_1}{1 - \sigma} + \delta \bar{V}'(\Gamma^*(H, 0) - \varepsilon)$$

$$\begin{aligned} &\leq \frac{\sigma - v_1}{1 - \sigma} + \delta v_2 && \text{from the claim above} \\ &< 0 && \text{because } v_1 > \sigma + (1 - \sigma)v_2 \text{ (Region 3)} \end{aligned}$$

This contradicts the fact that  $\Gamma^*(H, 0)$  is optimal is state  $(D, 0)$ . An analogous logic follows to show that  $\Gamma^*(L, 0) = \underline{\Gamma}_L$ .

### A.13. Proof of Proposition 6

Following the preliminary steps that we had in the main text, we can conclude that, in region 3, the policy function  $\Gamma^*(H, \Gamma)$  can be in three different shapes, which we denote by A1, A2, and A3:

A1) It is constant and equal to  $\underline{\Gamma}_H$  for each  $\Gamma \in \mathbf{\Gamma}$ .

A2) It increases with slope 1 up to the upper bound  $\bar{\Gamma}_H$ .

$$\Gamma^*(H, \Gamma) = \begin{cases} \Gamma + H - 1 & \text{if } \Gamma \leq 1 - \sigma H \\ \bar{\Gamma}_H & \text{if } \Gamma \geq 1 - \sigma H \end{cases}$$

A3) It increases with slope 1 up to  $\Gamma_1 \in (\underline{\Gamma}_H, \bar{\Gamma}_H)$  and stays constant afterwards.

$$\Gamma^*(H, \Gamma) = \begin{cases} \Gamma + H - 1 & \text{if } \Gamma \leq \Gamma_1 - H + 1 \\ \Gamma_1 & \text{if } \Gamma \geq \Gamma_1 - H + 1 \end{cases}$$

We already know the optimal policy satisfies  $\Gamma^*(L, \Gamma) = \underline{\Gamma}_L = 0$  for all  $\Gamma \leq 1 - L$ . For the same reason that we had three possible values for  $\Gamma^*(H, \Gamma)$ , the policy function  $\Gamma^*(L, \Gamma)$  can be in three different shapes in this region, which we denote by B1, B2, and B3:

B1) It is constant and equal to  $\underline{\Gamma}_L = 0$  for each  $\Gamma \in \mathbf{\Gamma}$ .

B2) It stays constant over  $\Gamma \leq 1 - L$ , and then it increases with slope 1 as long as it is feasible.

$$\begin{aligned} \text{--- Sub-case B2a: If } \sigma \geq 0.5, \text{ then } \Gamma^*(L, \Gamma) &= \begin{cases} 0 & \text{if } \Gamma \leq 1 - L \\ \Gamma - 1 + L & \text{if } \Gamma \geq 1 - L \end{cases} \\ \text{--- Sub-case B2b: If } \sigma < 0.5, \text{ then } \Gamma^*(L, \Gamma) &= \begin{cases} 0 & \text{if } \Gamma \leq 1 - L \\ \Gamma - 1 + L & \text{if } 1 - L \leq \Gamma \leq 1 - \sigma L \\ \bar{\Gamma}_L & \text{if } \Gamma \geq 1 - \sigma L \end{cases} \end{aligned}$$

B3) It stays constant over  $\Gamma \leq 1 - L$ , and then it increases with slope 1 up to a certain value  $\Gamma_2 \in (\underline{\Gamma}_L, \bar{\Gamma}_L)$  and stays constant afterwards.

$$\Gamma^*(L, \Gamma) = \begin{cases} 0 & \text{if } \Gamma \leq 1 - L \\ \Gamma + L - 1 & \text{if } \Gamma \in [1 - L, \Gamma_2 + 1 - L] \\ \Gamma_2 & \text{if } \Gamma \in [\Gamma_2 + 1 - L, (1 - \sigma)H] \end{cases}$$

In what follows, we analyze the three cases A1, A2, and A3 separately, and try to see under what circumstances these cases may arise in the optimal solution. This step involves in analyzing the coexistence of these cases with the three possibilities that we addressed for the function  $\Gamma^*(L, \Gamma)$ , i.e., cases B1, B2, and B3.

*Case A1:* In this case  $V(H, \Gamma)$  is constant, and hence  $V'(H, \Gamma) = 0, \forall \Gamma \in \mathbf{\Gamma}$ . Moreover, since it is optimal to set  $\Gamma^*(H, \Gamma)$  to its minimum value, i.e.,  $\underline{\Gamma}_H = H - 1$ , we have  $\frac{\partial_+ \tilde{V}(H, \Gamma, \underline{\Gamma}_H)}{\partial \Gamma'} \leq 0$ . It therefore yields, from Equation (24) that:

$$\frac{\sigma - v_1}{1 - \sigma} + v_2 + \delta(1 - k)V'_+(L, \underline{\Gamma}_H) \leq 0 \quad (25)$$

Let us first consider the case in which  $\frac{\sigma - v_1}{1 - \sigma} + v_2 + \delta(1 - k)v_2 \leq 0$ . In this case we must have  $\Gamma^*(L, \Gamma) = 0, \forall \Gamma \in \mathbf{\Gamma}$ , since  $V(D, \Gamma)$  is concave and  $V'(L, 0) = v_2$ . Then  $\Gamma^*(D, \Gamma)$  remains constant over  $\mathbf{\Gamma}$ , so Cases A1 and B1 can coexist.

Let us now consider the case where  $\frac{\sigma - v_1}{1 - \sigma} + v_2 + \delta(1 - k)v_2 > 0$ . We already know that  $\Gamma^*(L, \Gamma) = 0$ , and  $V'(L, \Gamma) = v_2$ , for each  $\Gamma \in [0, 1 - L]$ . Now, since  $\frac{\sigma - v_1}{1 - \sigma} + v_2 + \delta(1 - k)v_2 > 0$ , we must have  $\Gamma^*(L, \Gamma) = \Gamma + L - 1$  as long as  $\Gamma + L - 1 \leq 1 - L$ . This immediately rules out Case B1. We also know that  $\frac{\partial_+ \tilde{V}(H, \Gamma, \underline{\Gamma}_H)}{\partial \Gamma'} \leq 0$ , for each  $\Gamma \in \mathbf{\Gamma}$ . But we have:

$$\frac{\partial_+ \tilde{V}(L, \Gamma, L - 1)}{\partial \Gamma'} = \frac{\partial_+ \tilde{V}(H, \Gamma, \underline{\Gamma}_H)}{\partial \Gamma'} = \frac{\sigma - v_1}{\sigma} + v_2 + \delta \bar{V}'(1 - L) \leq 0, \quad \forall \Gamma \geq 1 - L.$$

Therefore,  $\Gamma^*(L, \Gamma) \leq \underline{\Gamma}_H = H - 1 = 1 - L$  without loss of generality, for each  $\Gamma \in \mathbf{\Gamma}$ . This yields  $\Gamma^*(L, \Gamma) = \min\{\bar{\Gamma}_L, 1 - L, \Gamma + L - 1\}$  for all  $\Gamma > 1 - L$ . In addition, note that, from Assumption 2, we have  $\bar{\Gamma}_H - 1 + L < 1 - L$ , so  $\Gamma - 1 + L < 1 - L$  for all  $\Gamma$ . Therefore, this is incompatible with Case B3. Hence, the only remaining possibility is B2. In order it to coexist with A1, we need to make sure that Equation (25) is satisfied. Since  $V'_+(L, 1 - L) = \frac{\sigma - v_1}{1 - \sigma} + v_2 + \delta(1 - k)v_2$ , this is equivalent to:

$$\frac{\sigma - v_1}{1 - \sigma}(1 + \delta(1 - k)) + v_2(1 + \delta(1 - k) + \delta^2(1 - k)^2) \leq 0.$$

The following claim summarizes these findings.

**CLAIM 6.** *In region 3, if the optimal solution  $\Gamma(H, \Gamma)$  satisfies Case A1, then there are two possibilities:*

- i)  $v_1 \geq \sigma + (1 - \sigma)v_2(1 + \delta(1 - k))$ , and  $\Gamma^*(L, \Gamma)$  satisfies Case B1.
- ii)  $v_1 \in \left[ \sigma + (1 - \sigma)v_2 \frac{1 + \delta(1 - k) + \delta^2(1 - k)^2}{1 + \delta(1 - k)}, \sigma + (1 - \sigma)v_2(1 + \delta(1 - k)) \right)$ , and  $\Gamma^*(L, \Gamma)$  satisfies Case B2.

*Case A2:* In this case, we have, for all  $\Gamma \geq \bar{\Gamma} + 1 - H$ ,  $\frac{\partial_- \tilde{V}(H, \Gamma, \bar{\Gamma}_H)}{\partial \Gamma'} \geq 0$ . Note that we have, in Case A2,  $\Gamma^*(H, \Gamma) = \Gamma_1$ , so  $\frac{\partial \Gamma^*(D, \Gamma)}{\partial \Gamma} = 0$  for  $\Gamma \geq \Gamma_1 - H + 1$ . Moreover,  $1 - H + \Gamma^*(H, \Gamma) \leq 1 - H + (1 - \sigma)H = 1 - \sigma H$  and  $\bar{\Gamma}_H = H - 1 \geq 1 - \sigma H$  per Assumption 2, which yields  $\frac{\partial \min\{\Gamma, 1 - D + \Gamma^*(D, \Gamma)\}}{\partial \Gamma} v_2$ . Per Equation (21), we obtain  $V'_-(H, \bar{\Gamma}_H) = 0$ . This implies that

$$\frac{\sigma - v_1}{1 - \sigma} + v_2 + \delta(1 - k)V'_-(L, \bar{\Gamma}_H) \geq 0$$

Due to the concavity of  $V$  (Proposition 3), it implies:

$$\frac{\sigma - v_1}{1 - \sigma} + v_2 + \delta(1 - k)V'_-(L, \Gamma) \geq 0, \quad \forall \Gamma \in \mathbf{\Gamma}.$$

Therefore, the optimal solution of  $\Gamma^*(L, \Gamma)$  must be of the form B2. Moreover, we must have  $\sigma \geq 0.5$ , because otherwise we would have  $1 - \sigma L < \bar{\Gamma}_H$  and  $V'(L, \bar{\Gamma}_H) = 0$ , which contradicts the above inequality  $v_1 > \sigma + (1 - \sigma)v_2$  in Region 3. Then the derivative of the value functions  $V(L, \Gamma)$ , and  $V(H, \Gamma)$  can be written as follows:

$$V'(H, \Gamma) = \begin{cases} \frac{\sigma - v_1}{1 - \sigma} + v_2 + \delta\bar{V}'(\Gamma + H - 1) & \text{if } \Gamma \leq 1 - \sigma H, & \text{Equation (24)} \\ 0 & \text{if } \Gamma \geq 1 - \sigma H. & \text{Equation (21)} \end{cases}$$

$$V'(L, \Gamma) = \begin{cases} v_2 & \text{if } \Gamma \leq 1 - L, & \text{Equation (22)} \\ \frac{\sigma - v_1}{1 - \sigma} + v_2 + \delta\bar{V}'(\Gamma + L - 1) & \text{if } \Gamma \geq 1 - L. & \text{Equation (24)} \end{cases}$$

Since  $\bar{\Gamma}_H + L - 1 = 1 - \sigma H$  and  $\bar{\Gamma}_H + L - 1 \leq 1 - L$  due to Assumption 2, we have:

$$V'_-(L, \bar{\Gamma}_H) = \frac{\sigma - v_1}{1 - \sigma} + v_2 + \delta k V'_-(H, 1 - \sigma H) + \delta(1 - k)v_2.$$

Moreover, we have:

$$\begin{aligned} V'_-(H, 1 - \sigma H) &= \frac{\sigma - v_1}{1 - \sigma} + v_2 + \delta(1 - k)V'_-(L, (1 - \sigma)H) \quad \text{because } V'_-(H, (1 - \sigma)H) = 0 \\ &= \frac{\sigma - v_1}{1 - \sigma} + v_2 + \delta(1 - k) \left( \frac{\sigma - v_1}{1 - \sigma} + v_2 + \delta k V'_-(H, 1 - \sigma H) + \delta(1 - k)v_2 \right) \\ &\quad \text{because } (1 - \sigma)H \geq 1 - L \text{ due to Assumption 1} \\ &= \frac{\sigma - v_1}{1 - \sigma} \left( \frac{1 + \delta(1 - k)}{1 - \delta^2(1 - k)k} \right) + v_2 \left( \frac{1 + \delta(1 - k) + \delta^2(1 - k)^2}{1 - \delta^2(1 - k)k} \right). \end{aligned}$$

Hence we can get:

$$V'_-(L, \bar{\Gamma}_H) = \frac{\sigma - v_1}{1 - \sigma} \left( \frac{1 + \delta k}{1 - \delta^2(1 - k)k} \right) + v_2 \left( \frac{1 + \delta}{1 - \delta^2(1 - k)k} \right).$$

Therefore the following comprises a necessary condition:

$$\frac{\sigma - v_1}{1 - \sigma} \left( \frac{1 + \delta(1 - k)}{1 - \delta^2(1 - k)k} \right) + v_2 \left( \frac{1 + \delta(1 - k) + \delta^2(1 - k)^2}{1 - \delta^2(1 - k)k} \right) \geq 0.$$

The following claim summarizes these findings regarding the case A2.

**CLAIM 7.** *In region 3, if the optimal solution  $\Gamma(H, \Gamma)$  satisfies Case A2 only if:*

- i)  $\sigma \geq 0.5$ , and  $\Gamma^*(L, \Gamma)$  satisfies B2a.*
- ii)  $v_1 < \sigma + (1 - \sigma)v_2 \frac{1 + \delta(1 - k) + \delta^2(1 - k)^2}{1 + \delta(1 - k)}$ .*

*Case A3:* As is in Case A2, for each  $\Gamma \in [\underline{\Gamma}_H, \bar{\Gamma}_H]$ ,  $\Gamma^*(H, \Gamma)$  is constant, and  $1 - H + \Gamma^*(H, \Gamma) \leq \Gamma$ ; therefore we have  $V'(H, \Gamma) = 0$  from Equation (21). Thus, if the optimal policy  $\Gamma^*(H, \Gamma)$  is of the form A3, then for all  $\Gamma > \Gamma_1 + 1 - H$  we must have:

$$\frac{\partial_- \tilde{V}(H, \Gamma, \Gamma_1)}{\partial \Gamma'} = \frac{\sigma - v_1}{1 - \sigma} + v_2 + \delta(1 - k)V'_-(L, \Gamma_1) \geq 0, \quad (26)$$

$$\frac{\partial_+ \tilde{V}(H, \Gamma, \Gamma_1)}{\partial \Gamma'} = \frac{\sigma - v_1}{1 - \sigma} + v_2 + \delta(1 - k)V'_+(L, \Gamma_1) \leq 0, \quad (27)$$

Moreover we have:

$$V'(H, \Gamma) = \begin{cases} \frac{\sigma - v_1}{1 - \sigma} + v_2 + \delta(1 - k)V'_L(\Gamma + H - 1) & \text{if } \Gamma \leq \Gamma_1 + 1 - H \\ 0 & \text{if } \Gamma \geq \Gamma_1 + 1 - H \end{cases}$$

We distinguish three cases.

A3-1) If  $\Gamma_1 \geq \bar{\Gamma}_L$ , and  $s \geq 0.5$ , then  $\Gamma^*(L, \Gamma)$  is of the form B2a. To see this, first note that, due to the optimality of  $\Gamma_1$  at  $\Gamma_1 - H + 1$ , we know that:

$$\frac{\sigma - v_1}{1 - \sigma} + v_2 + \delta \bar{V}'_-(\Gamma_1) \geq 0.$$

But then, for each  $\Gamma > 1 - L$ , setting the value  $\Gamma'$  equal to  $L - 1 + \Gamma$ , which is feasible, is optimal.

In other words:

$$\frac{\partial \tilde{V}(L, \Gamma, \Gamma + L - 1)}{\partial \Gamma'} = \frac{\sigma - v_1}{1 - \sigma} + v_2 + \delta \bar{V}'(\Gamma + L - 1) \geq 0, \quad \forall \Gamma \geq 1 - L.$$

This stems from the fact that  $\bar{V}'(\Gamma + L - 1) \geq \bar{V}'(\Gamma_1)$  since  $\Gamma + L - 1 < \Gamma_1$  and  $V(D, \Gamma)$  is a concave function of  $\Gamma$ . In this case, we also know that:

$$V'_L(\Gamma) = \begin{cases} v_2 & \text{if } \Gamma \leq 1 - L \\ \frac{\sigma - v_1}{1 - \sigma} + v_2 + \delta \bar{V}'(\Gamma + L - 1) & \text{if } \Gamma \geq 1 - L \end{cases}$$

Suppose  $\Gamma \leq \Gamma_1 + 1 - H$ , then we have

$$V'(H, \Gamma) = \frac{\sigma - v_1}{1 - \sigma} + v_2 + \delta(1 - k)V'(L, \Gamma + H - 1).$$

Moreover, since  $H - 1 = 1 - L$  (Assumption 2), we have  $\Gamma + H - 1 > 1 - L$ , and:

$$V'(L, \Gamma + H - 1) = \frac{\sigma - v_1}{1 - \sigma} + v_2 + \delta k V'(H, \Gamma) + \delta(1 - k)V'(L, \Gamma)$$

In addition, when  $\Gamma \leq \Gamma_1 + 1 - H$ , from Assumption 2 we must have  $\Gamma \leq 1 - L$  since  $\Gamma_1 + 1 - H < (1 - \sigma)H + 1 - H = 1 - \sigma H < H - 1 = 1 - L$ . Therefore,  $V'(L, \Gamma) = v_2$ . It comes:

$$V'(H, \Gamma) = \frac{\sigma - v_1}{1 - \sigma} + v_2 + \delta(1 - k) \left( \frac{\sigma - v_1}{1 - \sigma} + v_2 + \delta k V'(H, \Gamma) + \delta(1 - k)v_2 \right).$$

Therefore,

$$V'(H, \Gamma) = \begin{cases} \frac{\sigma - v_1}{1 - \sigma} \left( \frac{1 + \delta(1 - k)}{1 - \delta^2(1 - k)k} \right) + v_2 \left( \frac{1 + \delta(1 - k) + \delta^2(1 - k)^2}{1 - \delta^2(1 - k)k} \right) & \text{if } \Gamma \leq \Gamma_1 + 1 - H \\ 0 & \text{if } \Gamma \geq \Gamma_1 + 1 - H \end{cases}$$

$$V'(L, \Gamma) = \begin{cases} v_2 & \text{if } \Gamma \leq 1 - L \\ \frac{\sigma - v_1}{1 - \sigma} \left( \frac{1 + \delta k}{1 - \delta^2(1 - k)k} \right) + v_2 \left( \frac{1 + \delta}{1 - \delta^2(1 - k)k} \right) & \text{if } \Gamma \in [1 - L, \Gamma_1] \\ \frac{\sigma - v_1}{1 - \sigma} + v_2(1 + \delta(1 - k)) & \text{if } \Gamma \geq \Gamma_1 \end{cases}$$

Then the inequalities (26), and (27) can be written as:

$$\begin{aligned} \frac{\sigma - v_1}{1 - \sigma} + v_2 + \delta(1 - k) \left( \frac{\sigma - v_1}{1 - \sigma} \left( \frac{1 + \delta k}{1 - \delta^2(1 - k)k} \right) + v_2 \left( \frac{1 + \delta}{1 - \delta^2(1 - k)k} \right) \right) &\geq 0, \\ \frac{\sigma - v_1}{1 - \sigma} + v_2 + \delta(1 - k) \left( \frac{\sigma - v_1}{1 - \sigma} + v_2(1 + \delta(1 - k)) \right) &\leq 0. \end{aligned}$$

After rewriting these conditions we get:

$$\begin{aligned} \frac{\sigma - v_1}{1 - \sigma} \left( \frac{1 + \delta(1 - k)}{1 - \delta^2(1 - k)k} \right) + v_2 \left( \frac{1 + \delta(1 - k) + \delta^2(1 - k)^2}{1 - \delta^2(1 - k)k} \right) &\geq 0, \\ \frac{\sigma - v_1}{1 - \sigma} (1 + \delta(1 - k)) + v_2 (1 + \delta(1 - k) + \delta^2(1 - k)^2) &\leq 0. \end{aligned}$$

These conditions can hold at the same time if and only if both holds with equality. But this means that the value function  $V(H, \Gamma)$  is constant and hence  $\Gamma^*(H, \Gamma)$  could be increased up to  $\bar{\Gamma}_H$  (the value function would be unchanged). This, however, is already covered in case A2

A3-2) If  $\Gamma_1 \geq \bar{\Gamma}_L$ , and  $s < 0.5$ , then by proceeding as previously, we show that  $\Gamma^*(L, \Gamma)$  is of the form B2. But since not all the values of  $\Gamma + L - 1$  are included in  $[\underline{\Gamma}_L, \bar{\Gamma}_L]$  because  $\bar{\Gamma}_H - L + 1 > \bar{\Gamma}_L$  (since  $\sigma < 0.5$ ),  $\Gamma^*(L, \Gamma)$  is now of the form B2b. Note that  $V'(L, \Gamma) = 0$ , for every  $\Gamma \in [1 - \sigma L, (1 - \sigma)H]$ . This implies that we must have  $\Gamma_1 \leq 1 - \sigma L$ ; because otherwise  $V'(H, \Gamma_1 + 1 - H - \varepsilon) = \frac{\sigma - v_1}{1 - \sigma} + v_2 + \delta(1 - k)V'(L, \Gamma + H - 1)$  for  $\varepsilon > 0$  infinitesimally small. Then by following similar steps with the previous case, we reach:

$$V'(L, \Gamma) = \begin{cases} v_2 & \text{if } \Gamma \leq 1 - L \\ \frac{\sigma - v_1}{1 - \sigma} \left( \frac{1 + \delta k}{1 - \delta^2(1 - k)k} \right) + v_2 \left( \frac{1 + \delta}{1 - \delta^2(1 - k)k} \right) & \text{if } \Gamma \in [1 - L, \Gamma_1] \\ \frac{\sigma - v_1}{1 - \sigma} + v_2(1 + \delta(1 - k)) & \text{if } \Gamma \in [\Gamma_1, 1 - \sigma L] \\ 0 & \text{if } \Gamma \in [1 - \sigma L, (1 - \sigma)H] \end{cases}$$

Following the arguments of the previous section one can see that  $\Gamma_1 = 1 - \sigma L$ , and the following constitutes a necessary condition.

$$\frac{\sigma - v_1}{1 - \sigma} + v_2 + \delta(1 - k) \left( \frac{\sigma - v_1}{1 - \sigma} \left( \frac{1 + \delta k}{1 - \delta^2(1 - k)k} \right) + v_2 \left( \frac{1 + \delta}{1 - \delta^2(1 - k)k} \right) \right) \geq 0,$$

which is equivalent to:

$$\frac{\sigma - v_1}{1 - \sigma} \left( \frac{1 + \delta(1 - k)}{1 - \delta^2(1 - k)k} \right) + v_2 \left( \frac{1 + \delta(1 - k) + \delta^2(1 - k)^2}{1 - \delta^2(1 - k)k} \right) \geq 0.$$

A3-3) If  $\Gamma_1 \leq \bar{\Gamma}_L$ , then  $\Gamma_1$  is amongst the possible realizations of  $\Gamma^*(L, \Gamma)$ . However, in this case, the value of  $\Gamma + L - 1$ , the maximal value of  $\Gamma'$  in state  $(L, \Gamma)$ , is always less than  $\Gamma_1$ . To see this note that  $\Gamma + L - 1 \leq (1 - \sigma)H + L - 1 = 1 - \sigma H < H - 1 < \Gamma_1$  from Assumption 2. This suggests that the optimal policy  $\Gamma'(L, \Gamma)$  is linearly increasing with slope 1 when  $\Gamma > 1 - L$ , but it never reaches to the maximal value. The former point follows from the fact that  $\frac{\sigma - v_1}{1 - \sigma} + v_2 + \bar{V}'_-(\Gamma_1) \geq 0$ , and due to the concavity of  $\bar{V}$ ,  $\frac{\sigma - v_1}{1 - \sigma} + v_2 + \bar{V}'_-(\Gamma') \geq 0$  for each  $\Gamma' \leq \Gamma_1$ . In consequence, the optimal policy function  $\Gamma^*(L, \Gamma)$  is in form of B2a and  $\sigma \geq 0.5$ . We conclude as in the case where  $\Gamma_1 > \bar{\Gamma}$  and  $\sigma \geq 0.5$  (Case A3-1).

The following claim summarizes our analysis in Case A3. It points out that, if the policy function  $\Gamma^*(H, \Gamma)$  is of the form A3, then we can only have the second possibility (A3-2) out of the three.

CLAIM 8. *In region 3, if the optimal solution  $\Gamma(H, \Gamma)$  satisfies Case A3 only if:*

- i)  $\Gamma_1 = 1 - \sigma L$ .
- ii)  $\sigma < 0.5$  and  $\Gamma^*(L, \Gamma)$  is of the form B2b.
- iii)  $v_1 \leq \sigma + (1 - \sigma)v_2 \frac{1 + \delta(1-k) + \delta^2(1-k)^2}{1 + \delta(1-k)}$ ,

We conclude by putting Claims 6, 8, and 7 together.

## Appendix B: Proof on First-best and Surge Pricing Mechanisms

### B.1. Proof of Proposition 7

We have, by definition,  $\tilde{v}_1 = \sigma + (1 - \sigma)v_1$ . By construction, the following equality is satisfied:

$$\sigma D \left( 1 - \frac{(1 - \sigma)D - \Gamma'}{(1 - \sigma)D} (1 - \tilde{v}_1) \right) + ((1 - \sigma)D - \Gamma')\tilde{v}_1 = \sigma D + ((1 - \sigma)D - \Gamma')v_1$$

From Equation (10), Problem  $\hat{\mathcal{P}}$  with valuation parameter  $\tilde{v}_1$  is given by:

$$\begin{aligned} V(D, \Gamma) &= \max_{\Gamma' \in [\underline{\Gamma}_D, \bar{\Gamma}_D]} \left\{ \sigma D \left( 1 - \frac{(1 - \sigma)D - \Gamma'}{(1 - \sigma)D} (1 - \tilde{v}_1) \right) + ((1 - \sigma)D - \Gamma')\tilde{v}_1 + \min\{\Gamma, 1 - D - \Gamma'\}v_2 \right. \\ &\quad \left. + \delta(kV(H, \Gamma') + (1 - k)V(L, \Gamma')) \right\} \\ &= \max_{\Gamma' \in [\underline{\Gamma}_D, \bar{\Gamma}_D]} \sigma D + ((1 - \sigma)D - \Gamma')v_1 + \min\{\Gamma, 1 - D + \Gamma'\}v_2 + \delta(kV_f(H, \Gamma') + (1 - k)V_f(L, \Gamma)) \end{aligned}$$

From Equation (17), we obtain that the problem is equivalent to Problem  $\hat{\mathcal{P}}_f$ . The optimal policy is then obtained by applying the result of Proposition 6 by noting that  $\tilde{v}_1 \in (\sigma + (1 - \sigma)v_2, 1)$  for any  $v_1 \in (v_2, 1)$  and that the relationship between  $v_1$  and  $\tilde{v}_1$  is monotonic. Specifically, we have:

- $\tilde{v}_1 \in (\sigma + (1 - \sigma)v_2, \underline{v}_1]$ , if and only if  $v_1 \in \left( v_2, \frac{1 + \delta(1-k) + \delta^2(1-k)^2}{1 + \delta(1-k)} v_2 \right]$ .
- $\tilde{v}_1 \in (\underline{v}_1, \bar{v}_1]$ , if and only if  $v_1 \in \left( \frac{1 + \delta(1-k) + \delta^2(1-k)^2}{1 + \delta(1-k)} v_2, (1 + \delta(1 - k))v_2 \right]$ .
- $\tilde{v}_1 \in (\bar{v}_1, 1)$ , if and only if  $v_1 \in ((1 + \delta(1 - k))v_2, 1)$ .

## B.2. Proof of Proposition 8

From the formulation of  $\mathcal{P}_s$ , the Blackwell sufficiency conditions (monotonicity, and discounting) are clearly satisfied, hence a solution to Problem  $\widehat{\mathcal{P}}_s$  always exists. Moreover, since the quantity of rides provided  $x(D, \Gamma, p_s)$ , as well as the number of people transferred to the next period  $\Gamma'$  both stay constant over  $(0, v_2]$ ,  $(v_2, v_1]$ , and  $(v_1, 1]$ , the optimal price always satisfies  $p_s(D, \Gamma) \in \{1, v_1, v_2\}$ .

In a given state  $(D, \Gamma)$ , we assume, without loss of generality, that in case the platform is indifferent between charging  $p_s \in \{1, v_1\}$  and  $v_2$ , it chooses price  $p_s$ . We first prove that, if  $p_s^*(D, \Gamma) \in \{1, v_1\}$  for each  $(D, \Gamma)$ , then the optimal policy follows the one given in the proposition. We then show that this policy is applied if and only if  $v_2 \leq \max\{\sigma L, v_1 L\}$ .

We assume that  $p_s^*(D, \Gamma) \in \{1, v_1\}$  for each  $(D, \Gamma)$ . In this case,  $V_s(D, \Gamma)$  does *not* depend on the value of  $\Gamma$ . Therefore, the platform determines the price  $p_s$  by optimizing the current-period profit, i.e.  $p_s \times x(D, \Gamma, p_s)$ . Specifically, for each  $\Gamma$ ,  $p_s^*(H, \Gamma) = 1$  if  $\sigma H \geq v_1$ , and  $p_s^*(H, \Gamma) = v_1$  otherwise. Similarly,  $p_s^*(L, \Gamma) = 1$  if  $\sigma L \geq v_1 L$ , and  $p_s^*(L, \Gamma) = v_1$  otherwise. We therefore obtain the policy given in the proposition.

We now show that this policy is applied if and only if  $v_2 \leq \max\{\sigma L, v_1 L\}$ .

- Let us assume that  $p_s^*(D, \Gamma) \in \{1, v_1\}$  for each  $(D, \Gamma)$  and, by contradiction, that  $v_2 > \max\{\sigma L, v_1 L\}$ . Note that, since  $p_s^*(D, \Gamma) \in \{1, v_1\}$ ,  $V_s(D, \Gamma)$  does not depend on  $\Gamma$ . Let us consider  $\Gamma_0 > 1 - L$ . We have:

$$\text{If } p_s^*(L, \Gamma_0) = 1, \text{ then } V_s(L, \Gamma_0) = \sigma L + \delta(kV_s(H, (1 - \sigma)L) + (1 - k)V_s(L, (1 - \sigma)L))$$

$$\text{If } p_s^*(L, \Gamma_0) = v_1, \text{ then } V_s(L, \Gamma_0) = v_1 L + \delta(kV_s(H, 0) + (1 - k)V_s(L, 0))$$

However, if the platform deviates in the period under consideration by charging a price  $v_2$  without changing the policy in any of the subsequent periods, then its total expected discounted profit, denoted by  $\tilde{V}_s(L, \Gamma_0, v_2)$ , becomes:

$$\tilde{V}_s(L, \Gamma_0, v_2) = v_2 + \delta(kV_s(H, \Gamma') + (1 - k)V_s(L, \Gamma')) \quad \text{where } \Gamma' = (1 - \sigma)L \left[ 1 - \frac{1}{L + \Gamma_0} \right]$$

Since, by assumption,  $v_2 > \max\{\sigma L, v_1 L\}$ , and  $V_s(D, \Gamma)$  does not depend on  $\Gamma$ , we obtain that  $\tilde{V}_s(L, \Gamma_0, v_2) > V_s(L, \Gamma_0)$ . This contradicts the optimality of  $p_s^*(L, \Gamma_0)$ .

- Let us assume that  $v_2 \leq \max\{\sigma L, v_1 L\}$ . Note that setting  $p_s = v_2$  can bring a per-period profit of at most  $v_2$ . However, we have identified above a policy that yields a per-period profit of  $\max\{\sigma L, v_1 L\}$  in low-demand periods and of  $\max\{\sigma H, v_1\}$  in high-demand periods. Therefore, this policy dominates any policy that involves charging  $v_2$  in some states.